Topological Superconductivity

Annica Black-Schaffer

annica.black-schaffer@physics.uu.se

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Quantum Matter Theory

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Uppsala

University founded 1477

Ångström laboratory

Open postdoc positions in fall

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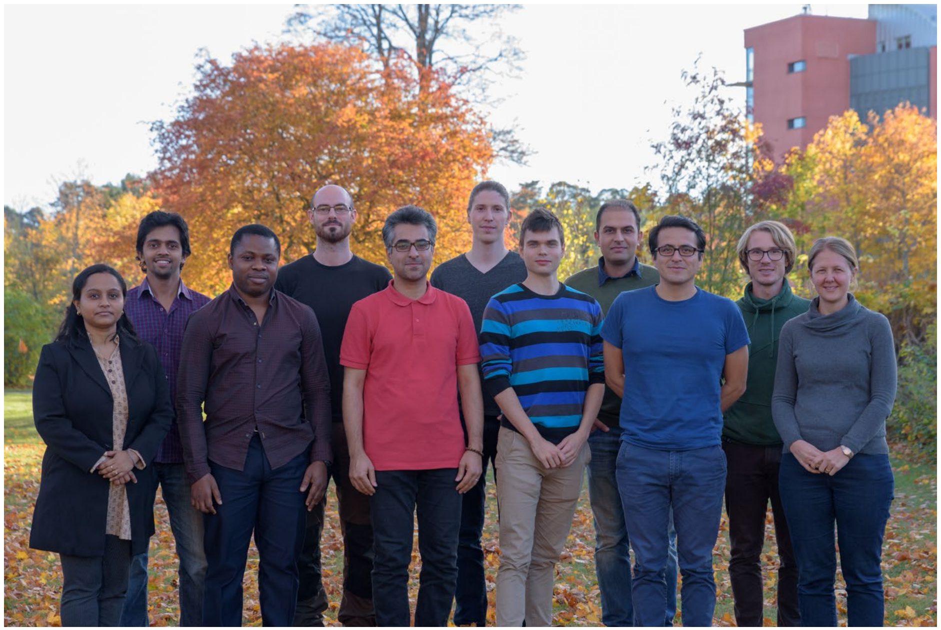

Kristofer Björnson, Tomas Löthman, Johann Schmidt, Ola Awoga, Suhas Nahas, Andreas Theiler, Lucas Casa, Florian Kayatz, Adrien Bouhon, Dushko Kuzmanovski, Lucia Komendova, Mahdi Mashkoori, Jorge Cayao, Fariborz Parhizgar, Christopher Triola, Paramita Dutta, Debmalya Chakraborty, Iman Mahyaeh, Patric Holmvall, Rodrigo Arouca, Tanay Nag, Quentin Marsal, Arnob Ghosh, Ankita Bhattacharya, Aksel Kobialka, Umberto Borla, Thanos Tsintzis, Raphael Teixeira, Anton Gregefalk

Collaborators:
A. Balatsky (Nordita), J. Linder (NTNU), J. Fransson (UU), K. Le Hur (Ecole Polytechnique), C. Honerkamp (Aachen), R. Aguado (Madrid), L. da Silva (Sao Paulo), M. Fogelström (Chalmers), S. Doniach (Stanford), C. Ast (MPI-FKF), F. Lombardi (Chalmers), A. Kantian (Heriot-Watt), Y. Tanaka (Nagoya), B. Sanyal (UU), H. Suderow, P. Burset (UA Madrid), Cristina Bena, Catherin Pepin (Paris-Saclay)

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Content

- Introduction to superconductivity
  - BCS, BdG, group theory

- Topological superconductivity
  - Chiral superconductors, e.g. *d*+i*d'*-wave superconductivity
  - "Spinless" p-wave superconductors → Majorana fermions

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Introduction to Superconductivity

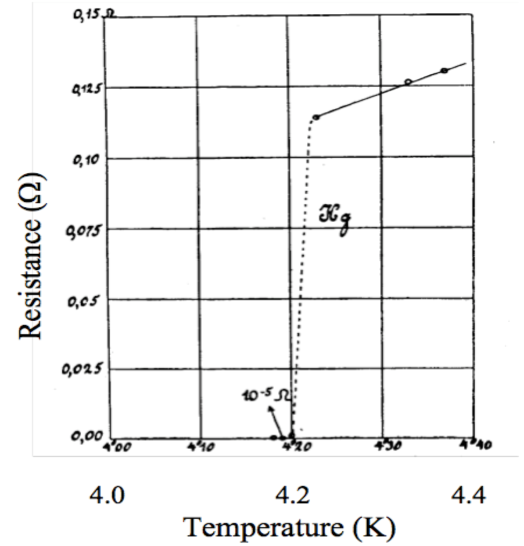
What is it?

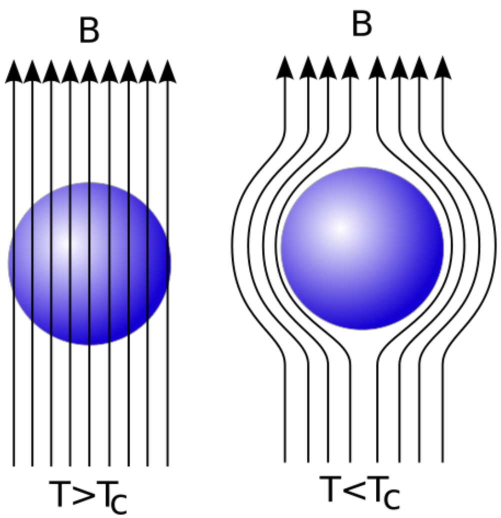
How do we describe it?

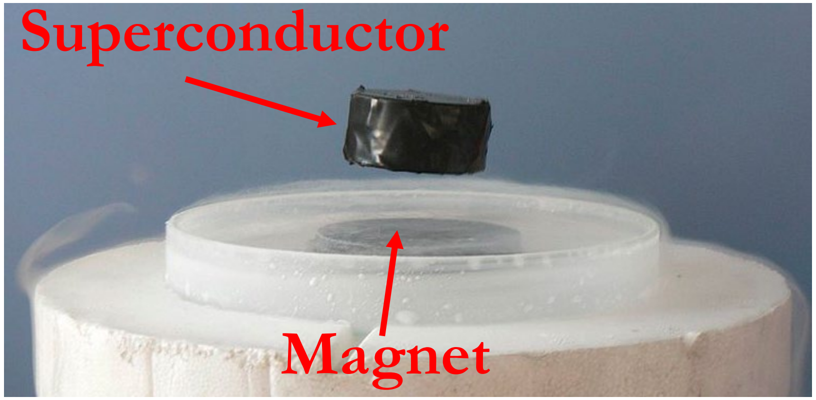
What is Superconductivity?

– Electric transport without resistance

– Meissner effect







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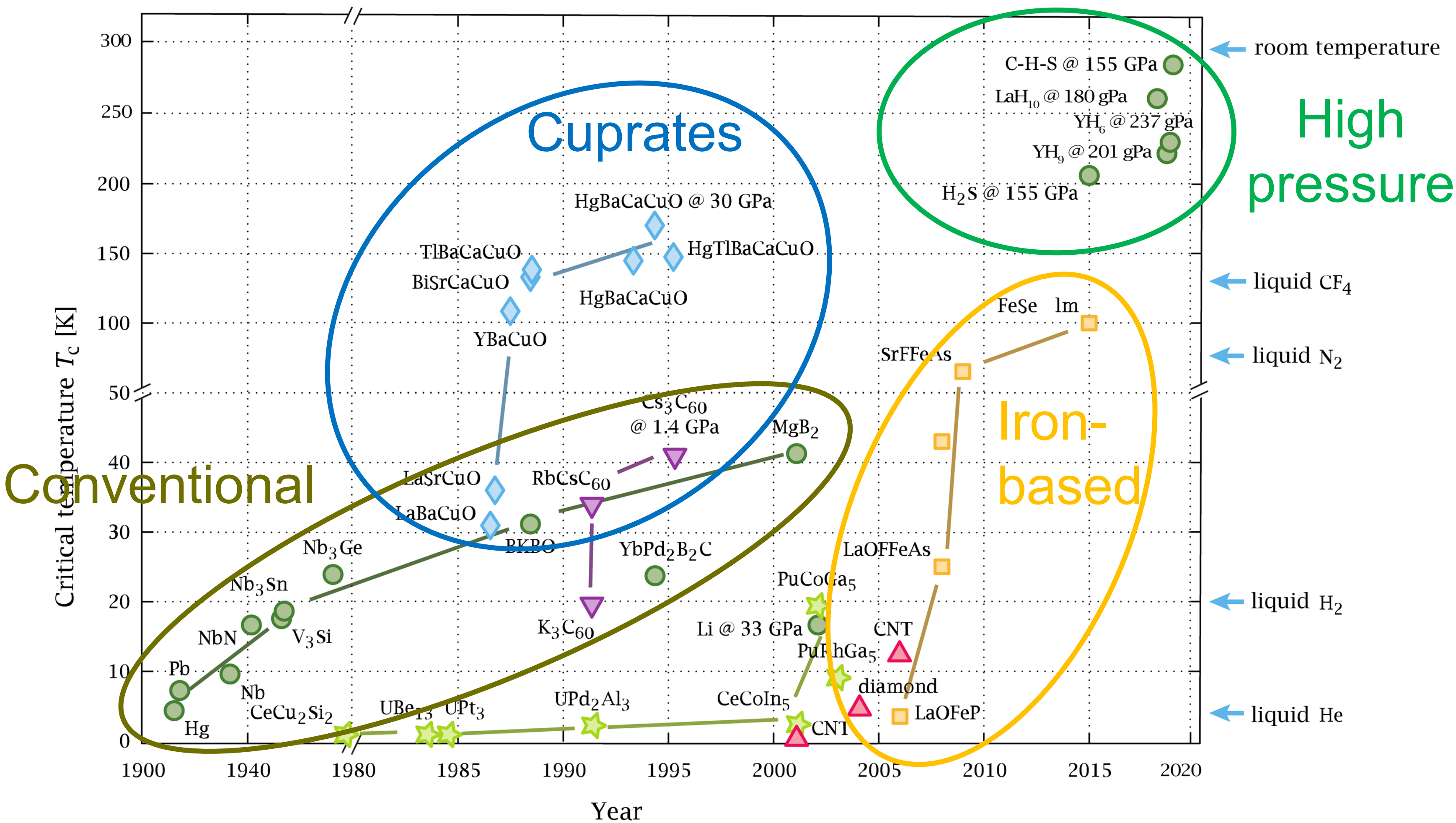
Superconductors



How?

- But how do electrons move without resistance?
  - All electrons in coherent quantum state with fixed phase (condensate)

$$\Psi = \Delta_0 e^{i\varphi}$$

- **Bardeen-Cooper-Schrieffer (BCS) theory**
  - Condensation of electron (Cooper) pairs (with fermionic wave function)
  - Many-body state, but possible to describe within mean-field theory

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BCS Hamiltonian

Pairing Hamiltonian:

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}\sigma} c^{\dagger}{}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} c^{\dagger}{}_{\mathbf{k}\uparrow} c^{\dagger}{}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

Kinetic (band) energy

Electron pairing

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -g_0/V, & (|\epsilon_{\mathbf{k}}| < \omega_D) \\ 0 & (\text{otherwise}) \end{cases}$$

Mean-field theory with $F_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$ (pair amplitude at $\mathbf{k}$)

$$\sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} [(c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} - F^{\dagger}_{\mathbf{k}}) + F^{\dagger}_{\mathbf{k}}][(c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} - F_{\mathbf{k}'}) + F_{\mathbf{k}'}] \approx$$

$$\sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} (F_{\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + F^{\dagger}_{\mathbf{k}} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} - F^{\dagger}_{\mathbf{k}} F_{\mathbf{k}'}]$$

Ignore fluctuations $(c^{\dagger} c^{\dagger} - F^{\dagger})(cc - F)$

Set **order parameter** $\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} F_{\mathbf{k}'} = \sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle$

$$\rightarrow \quad H_{\mathrm{MF}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} c^{\dagger}{}_{-\mathbf{k}\downarrow} c^{\dagger}_{\mathbf{k}\uparrow} + \text{H.c.} + \sum_{\mathbf{k}} \frac{V}{g_0} \Delta^{\dagger}_{\mathbf{k}} \Delta_{\mathbf{k}}$$

See e.g. Tinkham: Introduction to superconductivity

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Matrix Formulation (BdG)

Define the Nambu spinor $\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix}$ $\psi^{\dagger}{}_{\mathbf{k}} = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right)$

$$\rightarrow \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}\left(c^{\dagger}{}_{\mathbf{k}\uparrow}c_{\mathbf{k}\uparrow} - c_{-\mathbf{k}\downarrow}c^{\dagger}{}_{-\mathbf{k}\downarrow} + 1\right) = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right)\begin{bmatrix} \epsilon_{\mathbf{k}} & 0 \\ 0 & -\epsilon_{\mathbf{k}} \end{bmatrix}\begin{pmatrix} c_{\mathbf{k}\uparrow} \\ \bar{c}^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix} + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}$$

TRS: $\varepsilon_{\mathrm{k}} = \varepsilon_{-\mathrm{k}}$

constant

$$\rightarrow \epsilon_{\mathbf{k}} \sum_{\sigma} c^{\dagger}{}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + \left[\bar{\Delta}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} + c^{\dagger}{}_{\mathbf{k}\uparrow}c^{\dagger}{}_{-\mathbf{k}\downarrow}\Delta\right] = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right)\begin{bmatrix} \epsilon_{\mathbf{k}} & \Delta \\ \bar{\Delta} & -\epsilon_{\mathbf{k}} \end{bmatrix}\begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix}$$

Bogliubov-de Gennes (BdG) formulation

2x2 matrix problem

→ Solve by finding eigenvalues and vectors

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Eigenstates = Quasiparticles

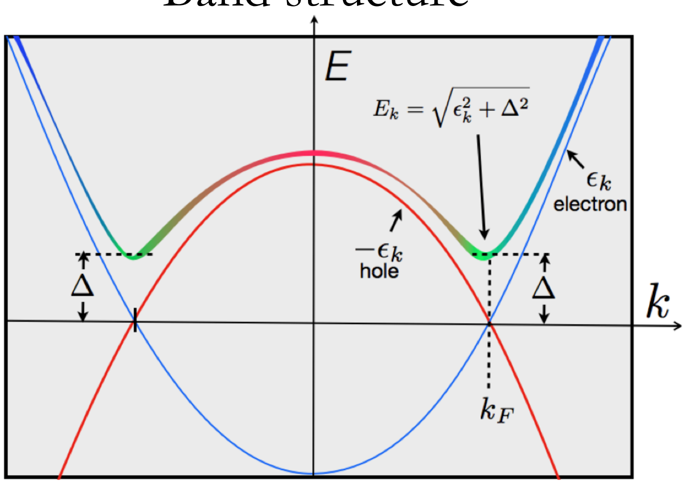
QP energies (eigenvalues): $E_{\mathbf{k}} = \pm\sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$

QP operators (eigenvectors): $\begin{cases} a^\dagger{}_{\mathbf{k}\uparrow} = \psi^\dagger{}_{\mathbf{k}} \cdot \begin{pmatrix} u_{\mathbf{k}} \\ \mathrm{v}_{\mathbf{k}} \end{pmatrix} = c^\dagger{}_{\mathbf{k}\uparrow} u_{\mathbf{k}} + c_{-\mathbf{k}\downarrow} \mathrm{v}_{\mathbf{k}} \\ a_{-\mathbf{k}\downarrow} = \psi^\dagger{}_{\mathbf{k}} \cdot \begin{pmatrix} -\mathrm{v}^*_{\mathbf{k}} \\ u^*_{\mathbf{k}} \end{pmatrix} = c_{-\mathbf{k}\downarrow} u^*_{\mathbf{k}} - c^\dagger{}_{\mathbf{k}\uparrow} \mathrm{v}^*_{\mathbf{k}} \end{cases}$

Bogoliubov tranformation

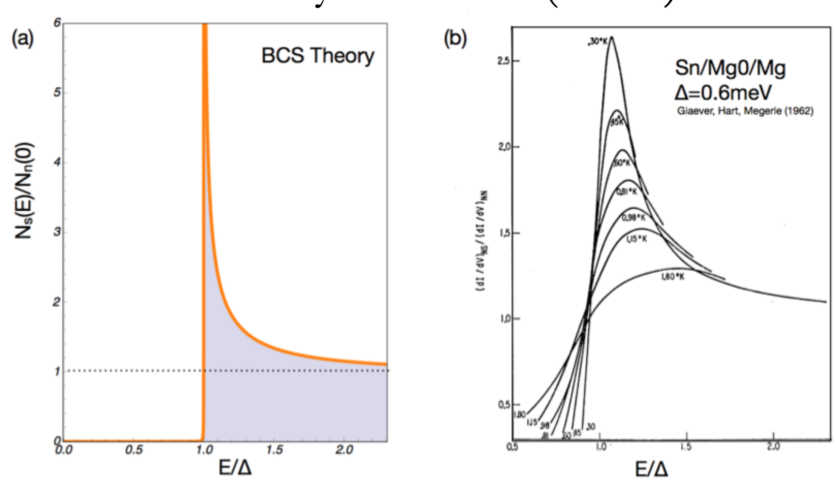
$$u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left[1 + \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right]} \qquad \mathrm{v}_{\mathbf{k}} = \sqrt{\frac{1}{2}\left[1 - \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right]}$$

Band structure



P. Coleman: Introduction to Many Body Physics

Density of states (DOS)



Superconducting Order

Self-consistent order parameter:

$$\Delta_{\mathbf{k}} = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} u_{\mathbf{k}'} v^*_{\mathbf{k}'} \langle a^\dagger_{\mathbf{k}'\uparrow} a_{\mathbf{k}'\uparrow} + a^\dagger_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\downarrow} - 1 \rangle$$

$$\langle a^\dagger_{\mathbf{k}'\sigma} a_{\mathbf{k}'\sigma} \rangle = (1 + e^{E_{\mathbf{k}'}/k_B T})^{-1}$$ Fermi-Dirac distribution

Generalized order: fermionic, odd under particle exchange:

$$\Delta_{\alpha\beta}(\mathbf{k}) = -\Delta_{\beta\alpha}(-\mathbf{k})$$

$$\Delta_{\alpha\beta}(\mathbf{k}) = \Delta e^{i\varphi} \eta(\mathbf{k}) \chi_{\alpha\beta}$$

($\eta(\mathbf{k})$: orbital; $\chi_{\alpha\beta}$: spin)

$$\chi_{\alpha\beta} \rightarrow \begin{cases} \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) & (S=0) \\ |\uparrow\uparrow\rangle & (S=1, S_z=1) \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) & (S=1, S_z=0) \\ |\downarrow\downarrow\rangle & (S=1, S_z=-1) \end{cases}$$

$S=0$ → η even function in $\mathbf{k}$

$S=1$ ($S_z = 1, 0, -1$) → η odd function in $\mathbf{k}$

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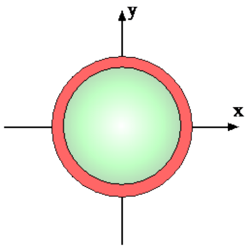
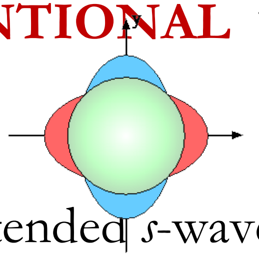
Superconducting Pairing

$V_{\mathbf{k},\mathbf{k}'}$ (and the band structure) determine the pairing symmetry, but often very hard to determine

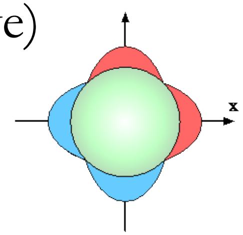
- Lattice fluctations (phonon): spin-singlet *s*-wave **CONVENTIONAL**



- Cuprate high-Tc superconductors: spin-singlet *d*-wave



- Antiferromagnetic spin fluctuations: spin-singlet *d*-wave (extended *s*-wave)
- Ferromagnetic spin fluctuations: spin-triplet *p*-wave



- Strong on-site repulsion (Heisenberg interaction): spin-singlet *d*-wave
- …

Can we determine the possible pairing symmetries in a material without knowing $V_{\mathbf{k},\mathbf{k}'}$?

Yes, by a general group theory analysis

See e.g. Sigrist and Ueda, RMP **63**, 239 (1991)

General Solution

4x4 BdG equation: $\tilde{\mathcal{H}} = \mathbf{a}_{\mathbf{k}}^{\dagger} \begin{pmatrix} \varepsilon(\mathbf{k}) & \hat{\Delta}(\mathbf{k}) \\ \hat{\Delta}^{\dagger}(\mathbf{k}) & -\varepsilon(\mathbf{k}) \end{pmatrix} \mathbf{a}_{\mathbf{k}}$

$$\left[\mathbf{a}_{\mathbf{k}} = (a_{\mathbf{k}\uparrow}, a_{\mathbf{k}\downarrow}, a^{\dagger}_{-\mathbf{k}\uparrow}, a^{\dagger}_{-\mathbf{k}\downarrow})^{\mathsf{T}} \right]$$

Self-consistency equation: $v\Delta_{s_1 s_2}(\mathbf{k}) = -\sum_{s_3 s_4} \langle V_{s_2 s_1 s_3 s_4}(\mathbf{k},\mathbf{k}')\Delta_{s_3 s_4}(\mathbf{k}') \rangle_{\mathbf{k}'}$

- Linear equation close to T_c
  - Largest eigenvalue gives T_c
  - Eigenfunction (Δ) belongs to irreducible representation (irrep) of symmetry group

→ Possible SC symmetries belong to **irreps of symmetry group of H**

→ SC state always breaks U(1), can also break

- Crystal lattice, spin-rotation, time-reversal, … symmetries

Basis Gap Functions: D_{4h}

- D_{4h} = tetragonal symmetry (cuprates with $k_z = 0$)

Irreducible representation Γ	Basis function	
	(a) Spin-singlet	
Γ_1^+	$\psi(\Gamma_1^+;\mathbf{k})=1,\ k_x^2+k_y^2,\ k_z^2$	*s*-wave, extended *s*-wave
Γ_2^+	$\psi(\Gamma_2^+;\mathbf{k})=k_xk_y(k_x^2-k_y^2)$	
Γ_3^+	$\psi(\Gamma_3^+;\mathbf{k})=k_x^2-k_y^2$	*d*(x^2-y^2)-wave
Γ_4^+	$\psi(\Gamma_4^+;\mathbf{k})=k_xk_y$	*d*(xy)-wave
Γ_5^+	$\psi(\Gamma_5^+,1;\mathbf{k})=k_xk_z$	
	$\psi(\Gamma_5^+,2;\mathbf{k})=k_yk_z$	
	(b) Spin-triplet	
Γ_1^-	$\mathbf{d}(\Gamma_1^-;\mathbf{k})=\hat{\mathbf{x}}k_x+\hat{\mathbf{y}}k_y,\ \hat{\mathbf{z}}k_z$	
Γ_2^-	$\mathbf{d}(\Gamma_2^-;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$	
Γ_3^-	$\mathbf{d}(\Gamma_3^-;\mathbf{k})=\hat{\mathbf{x}}k_x-\hat{\mathbf{y}}k_x$	
Γ_4^-	$\mathbf{d}(\Gamma_4^-;\mathbf{k})=\hat{\mathbf{x}}k_y+\hat{\mathbf{y}}k_x$	
Γ_5^-	$\mathbf{d}(\Gamma_5^-,1;\mathbf{k})=\hat{\mathbf{x}}k_z,\ \hat{\mathbf{z}}k_x$	*p*(x)- and *p*(y)-wave degenerate
	$\mathbf{d}(\Gamma_5^-,2;\mathbf{k})=\hat{\mathbf{y}}k_z,\ \hat{\mathbf{z}}k_y$	

Sigrist and Ueda, RMP **63**, 239 (1991)

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Basis Gap Functions: D_{6h}

- D_{6h} = hexagonal symmetry (graphene, Bi_2Se_3 TIs with $k_z = 0$,)

Irreducible representation Γ	Basis functions	
	(a) Spin-singlet	
Γ_1^+	$\psi(\Gamma_1^+;\mathbf{k})=1,\ k_x^2+k_y^2,\ k_z^2$	*s*-wave, extended *s*-wave
Γ_2^+	$\psi(\Gamma_2^+;\mathbf{k})=k_xk_y(k_x^2-3k_y^2)(k_y^2-3k_x^2)$	
Γ_3^+	$\psi(\Gamma_3^+;\mathbf{k})=k_zk_x(k_x^2-3k_y^2)$	
Γ_4^+	$\psi(\Gamma_4^+;\mathbf{k})=k_zk_y(k_y^2-3k_x^2)$	
Γ_5^+	$\psi(\Gamma_5^+,1;\mathbf{k})=k_xk_z$ $\psi(\Gamma_5^+,2;\mathbf{k})=k_yk_z$	
Γ_6^+	$\psi(\Gamma_6^+,1;\mathbf{k})=k_x^2-k_y^2$ $\psi(\Gamma_6^+,2;\mathbf{k})=2k_xk_y$	*d*(x^2-y^2)-wave and *d*(xy)-wave degenerate
	(b) Spin-triplet	
Γ_1^-	$\mathbf{d}(\Gamma_1^-;\mathbf{k})=\hat{\mathbf{x}}k_x+\hat{\mathbf{y}}k_y,\hat{\mathbf{z}}k_z$	
Γ_2^-	$\mathbf{d}(\Gamma_2^-;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$	
Γ_3^-	$\mathbf{d}(\Gamma_3^-;\mathbf{k})=\hat{\mathbf{z}}k_x(k_x^2-3k_y^2),$ $k_z[(k_x^2-k_y^2)\hat{\mathbf{x}}-2k_xk_y\hat{\mathbf{y}}]$	
Γ_4^-	$\mathbf{d}(\Gamma_4^-;\mathbf{k})=\hat{\mathbf{z}}k_y(k_y^2-3k_x^2),$ $k_z[(k_y^2-k_x^2)\hat{\mathbf{y}}-2k_xk_y\hat{\mathbf{x}}]$	
Γ_5^-	$\mathbf{d}(\Gamma_5^-,1;\mathbf{k})=\hat{\mathbf{x}}k_z,\hat{\mathbf{z}}k_x$ $\mathbf{d}(\Gamma_5^-,2;\mathbf{k})=\hat{\mathbf{y}}k_z,\hat{\mathbf{z}}k_y$	
Γ_6^-	$\mathbf{d}(\Gamma_6^-,1;\mathbf{k})=\hat{\mathbf{x}}k_x-\hat{\mathbf{y}}k_y$ $\mathbf{d}(\Gamma_6^-,2;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$	

Sigrist and Ueda, RMP **63**, 239 (1991)

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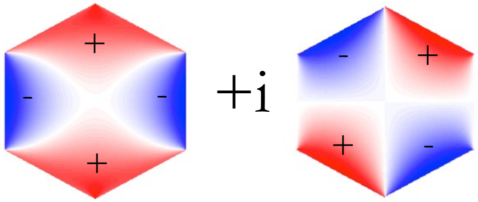
Multiple Order Parameters

SC state highly unconventional if multiple components at T_c

- Two-dimensional irreps often gives $\Delta_1 + i\Delta_2$ at $T < T_c$
  - Only combination with full gap → Highest energy gain
  - Singlet $d(x^2-y^2)+id(xy)$-wave for hexagonal/trigonal lattices



  - Triplet ($m_z = 0$) $p(x)+ip(y)$-wave for square lattices

Chiral superconductors

full energy gap, break time-reversal symmetry (TRS), topological

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Introduction to Superconductivity

What is it?

- A charged superfluid of Cooper pairs (2 electrons) with fermionic character
- Cooper pairs formed by effective attractive interaction

How do we describe it?

- BCS theory (mean-field theory of condensation)
- BdG matrix formalism
- Symmetry of order parameter (group theory)

Topological Superconductivity

Introduction

Chiral superconductors

Spin-singlet d+id'-wave (spin-triplet p+ip'-wave) superconductors

Spinless superconductors

Majorana fermions

Engineered systems

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Topological Superconductivity

Introduction

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Spin-singlet *d*+i*d'*-wave (spin-triplet *p*+i*p'*-wave) superconductors

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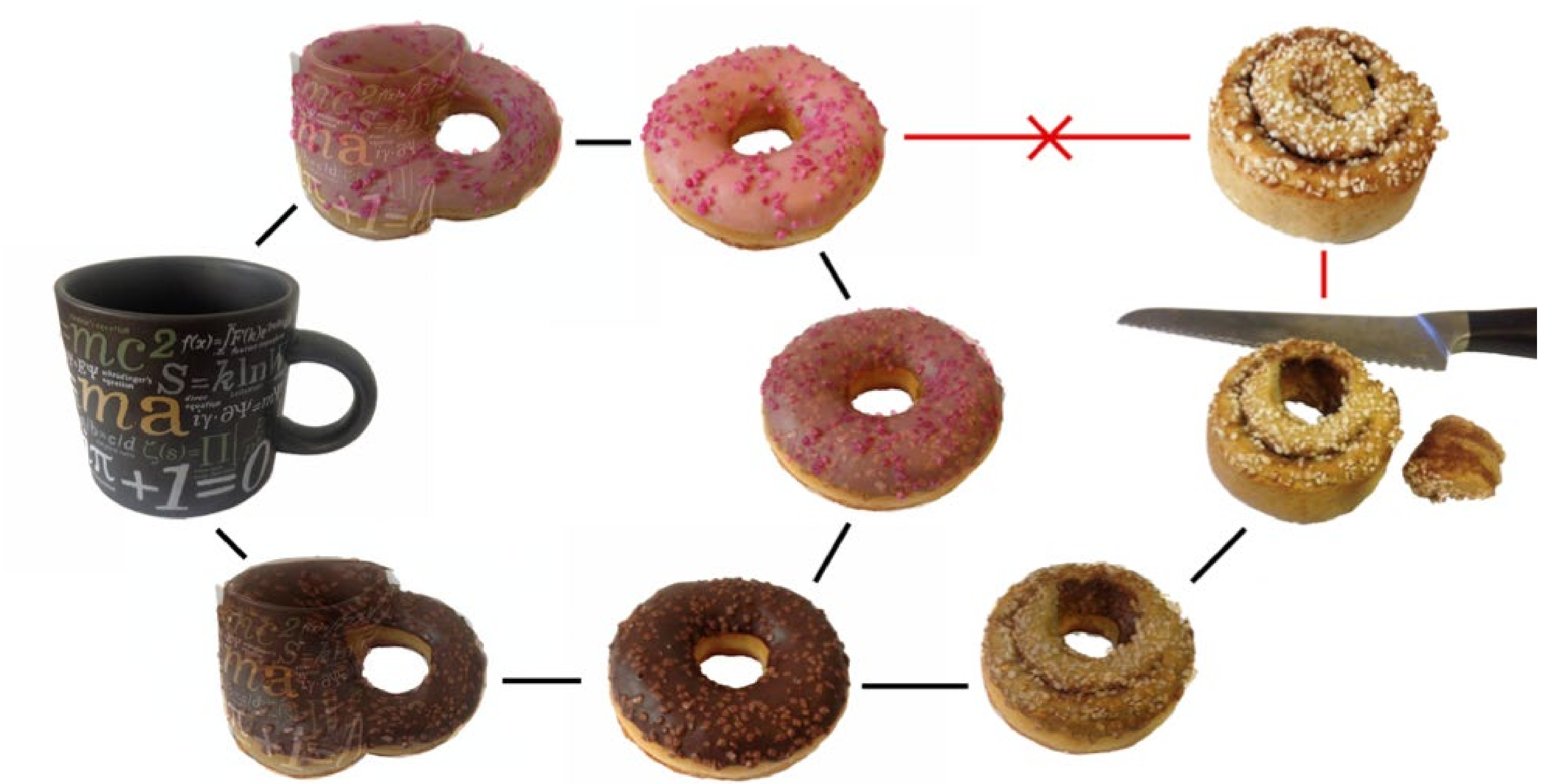
Topology



Topologically speaking: coffee cup = donut ≠ bun

1 hole — 1 hole — 0 hole

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Classification

All forms of matter can be classified according to the symmetry they break

→ Local order parameter

Except topological matter

Topological insulators: 2005 (quantum Hall effect: 1980)

- Ordered but no symmetry breaking
- Topology of energy bands (or wave functions)

→ Global topological order

Topological superconductors have both!

Topological Matter

Topological states of matter have

- **Bulk topological invariant**
  - Number classifying the topological class
  - Only changes with the bulk gap closing
- **Protected boundary states**
  - At any boundary to other topological or trivial region (vacuum, normal metal, *s*-wave SC = trivial topological order)

Bulk-boundary correspondence

# of boundary states = change in topological invariant at boundary

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Topological Classification (gapped systems)

Non-interacting (single-particle) insulators and superconductors: 10-fold way

		TRS (time-reversal)	PHS (particle-hole)	SLS (sublattice (chiral))	$d=1$	$d=2$	$d=3$
					Topological invariants		
Standard (Wigner-Dyson)	A (unitary)	0	0	0	-	$\mathbb{Z}$	-
	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Chiral (sublattice)	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
	CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
	C	0	−1	0	-	$\mathbb{Z}$	-
	DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	+1	−1	1	-	-	$\mathbb{Z}$

Schnyder *et al.*, PRB **78**, 195125 (2008)

Superconductors

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AZ class	SU(2)	TRS	Examples in two dimensions
D	×	×	Spinless chiral $(p \pm ip)$ wave
DIII	×	○	Superposition of $(p+ip)$ and $(p-ip)$ waves
A	△	×	Spinful chiral $(p \pm ip)$ wave
AIII	△	○	Spinful p_x or p_y wave
C	○	×	$(d \pm id)$ wave
CI	○	○	$d_{x^2-y^2}$ or d_{xy} wave

		TRS	PHS	SLS	d=1	d=2	d=3
Standard	A (unitary)	0	0	0	-	Z	-
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	Z_2	Z_2
Chiral	AIII (chiral unitary)	0	0	1	Z	-	Z
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	Z	-	-
	CII (chiral symplectic)	−1	−1	1	Z	-	Z_2
BdG	D	0	+1	0	Z_2	Z	-
	C	0	−1	0	-	Z	-
	DIII	−1	+1	1	Z_2	Z_2	Z
	CI	+1	−1	1	-	-	Z

Spinless *p*+i*p*'-wave in 1D → BDI because effective TRS

Schnyder *et al.*, PRB **78**, 195125 (2008)

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Engineered systems

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d-wave SC from Strong Repulsion

Strong Coulomb repulsion, antiferromagnetic correlations

(e.g. Hubbard model near half-filling)

→ Spin-singlet pairing

→ Double electron occupation unfavorable

→ No *s*-wave pairing

→ Spin-singlet *d*-wave pairing

(best state = least number of nodes)

2D hexagonal lattice →

Spin-singlet $d(x^2-y^2)+id(xy)$ pairing

(Only combination with full energy gap)

Irreducible representation Γ	Basis functions
	(a)
Γ_1^+	$\psi(\Gamma_1^+;\mathbf{k})=1,\ k_x^2+k_y^2,\ k_z^2$
Γ_2^+	$\psi(\Gamma_2^+;\mathbf{k})=k_xk_y(k_x^2-3k_y^2)(k_y^2-3k_x^2)$
Γ_3^+	$\psi(\Gamma_3^+;\mathbf{k})=k_zk_x(k_x^2-3k_y^2)$
Γ_4^+	$\psi(\Gamma_4^+;\mathbf{k})=k_zk_y(k_y^2-3k_x^2)$
Γ_5^+	$\psi(\Gamma_5^+,1;\mathbf{k})=k_xk_z$ $\psi(\Gamma_5^+,2;\mathbf{k})=k_yk_z$
Γ_6^+	$\psi(\Gamma_6^+,1;\mathbf{k})=k_x^2-k_y^2$ $\psi(\Gamma_6^+,2;\mathbf{k})=2k_xk_y$ $d(x^2-y^2)+id(xy)$
	(b)
Γ_1^-	$\mathbf{d}(\Gamma_1^-;\mathbf{k})=\hat{\mathbf{x}}k_x+\hat{\mathbf{y}}k_y,\hat{\mathbf{z}}k_z$
Γ_2^-	$\mathbf{d}(\Gamma_2^-;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$
Γ_3^-	$\mathbf{d}(\Gamma_3^-;\mathbf{k})=\hat{\mathbf{z}}k_x(k_x^2-3k_y^2),$ $k_z[(k_x^2-k_y^2)\hat{\mathbf{x}}-2k_xk_y\hat{\mathbf{y}}]$
Γ_4^-	$\mathbf{d}(\Gamma_4^-;\mathbf{k})=\hat{\mathbf{z}}k_y(k_y^2-3k_x^2),$ $k_z[(k_y^2-k_x^2)\hat{\mathbf{y}}-2k_xk_y\hat{\mathbf{x}}]$
Γ_5^-	$\mathbf{d}(\Gamma_5^-,1;\mathbf{k})=\hat{\mathbf{x}}k_z,\hat{\mathbf{z}}k_x$ $\mathbf{d}(\Gamma_5^-,2;\mathbf{k})=\hat{\mathbf{y}}k_z,\hat{\mathbf{z}}k_y$
Γ_6^-	$\mathbf{d}(\Gamma_6^-,1;\mathbf{k})=\hat{\mathbf{x}}k_x-\hat{\mathbf{y}}k_y$ $\mathbf{d}(\Gamma_6^-,2;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$

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Chiral *d*+i*d'* SCs?

- Superconducting graphene
- SrPtAs
- Na_xCoO_2 • yH_2O
- β-MNCl
- κ-$(BEDT\text{-}TTF)_2X$
- (111) bilayer $SrIrO_3$
- $In_3Cu_2VO_9$
- Twisted (~45º) cuprate bilayers

See also review: ABS and Honerkamp JPCM **26**, 423201 (2014)

Bulk and Edge Properties

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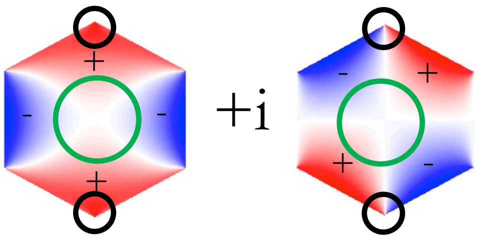
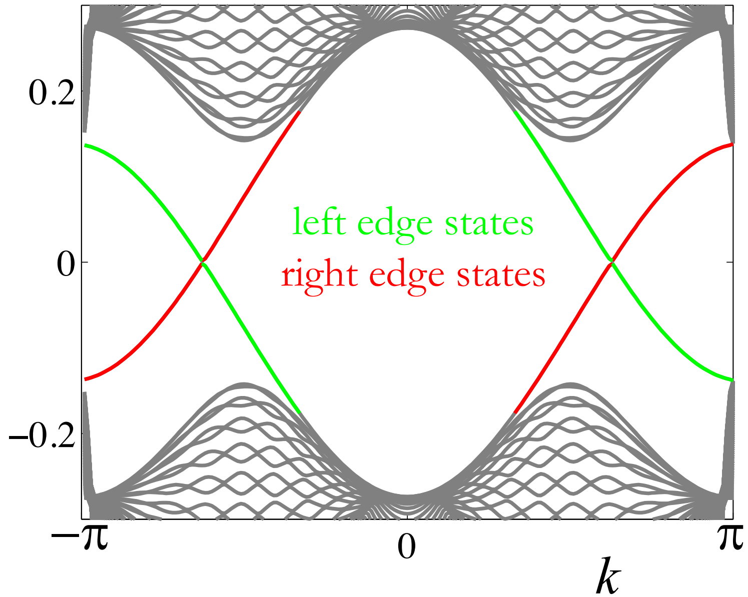
- Fully gapped bulk $E_{\mathrm{QP}}(\mathbf{k}) = \sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}$



- Two chiral (co-propagating) edge states per edge



ABS, PRL **109**, 197001 (2012)

Topological Invariant

d+id'-wave SC breaks TRS → Chern number invariant

$$\mathcal{N} = \frac{1}{4\pi}\int_{\mathrm{BZ}} \mathrm{d}^2k\, \hat{\mathbf{m}} \cdot \left(\frac{\partial \hat{\mathbf{m}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{m}}}{\partial k_y}\right)$$

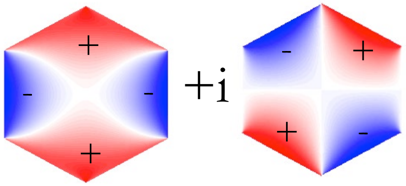
Skyrmion number

$$\hat{\mathbf{m}} = \frac{1}{\sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}} \begin{pmatrix} \mathrm{Re}\,\Delta(\mathbf{k}) \\ \mathrm{Im}\,\Delta(\mathbf{k}) \\ \varepsilon(\mathbf{k}) \end{pmatrix}$$

Counts unit sphere area spanned by **m** as **k** covers the BZ

Bottom of band: **m** ~ -**z**
Top of band: **m** ~ **z**
→ Non-zero N iff Δ has finite winding along lines of constant ε

d+id'-wave winds twice around Γ → **|N| = 2**
→ **2 chiral edge states**



ABS and Honerkamp JPCM **26**, 423201 (2014)

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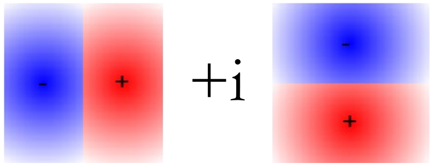
Chiral *p*+i*p* SCs

Spin-triplet *p*(x)+i*p*(y)-wave spin-triplet, $\mathbf{d} = (0, 0, k_x + ik_y)$

- Fully gapped in the bulk $E_{\mathrm{QP}}(\mathbf{k}) = \sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}$



p(x) +i *p*(y)

- Breaks TRS → finite Chern number/Skyrmion winding

$$\mathcal{N} = \frac{1}{4\pi}\int_{\mathrm{BZ}} \mathrm{d}^2k\, \hat{\mathbf{m}} \cdot \left(\frac{\partial \hat{\mathbf{m}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{m}}}{\partial k_y}\right)$$

$$\hat{\mathbf{m}} = \frac{1}{\sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}} \begin{pmatrix} \mathrm{Re}\,\Delta(\mathbf{k}) \\ \mathrm{Im}\,\Delta(\mathbf{k}) \\ \varepsilon(\mathbf{k}) \end{pmatrix}$$

p+i*p*'-wave winds once around Γ → **|N| = 1**

→ **One chiral edge state per edge**

Different Chiralities

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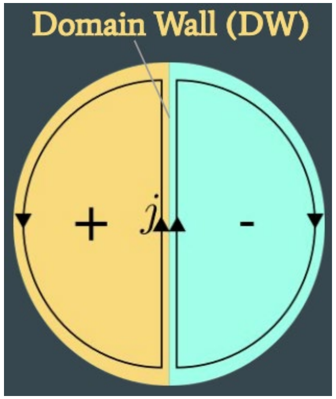
- Bulk: *d*±i*d'*-wave states are degenerate
  → One solution chosen spontaneously
- Domain walls: Δ**N** = 2 **N** → 2**N** DW boundary states

p(x)+i*p*(y):

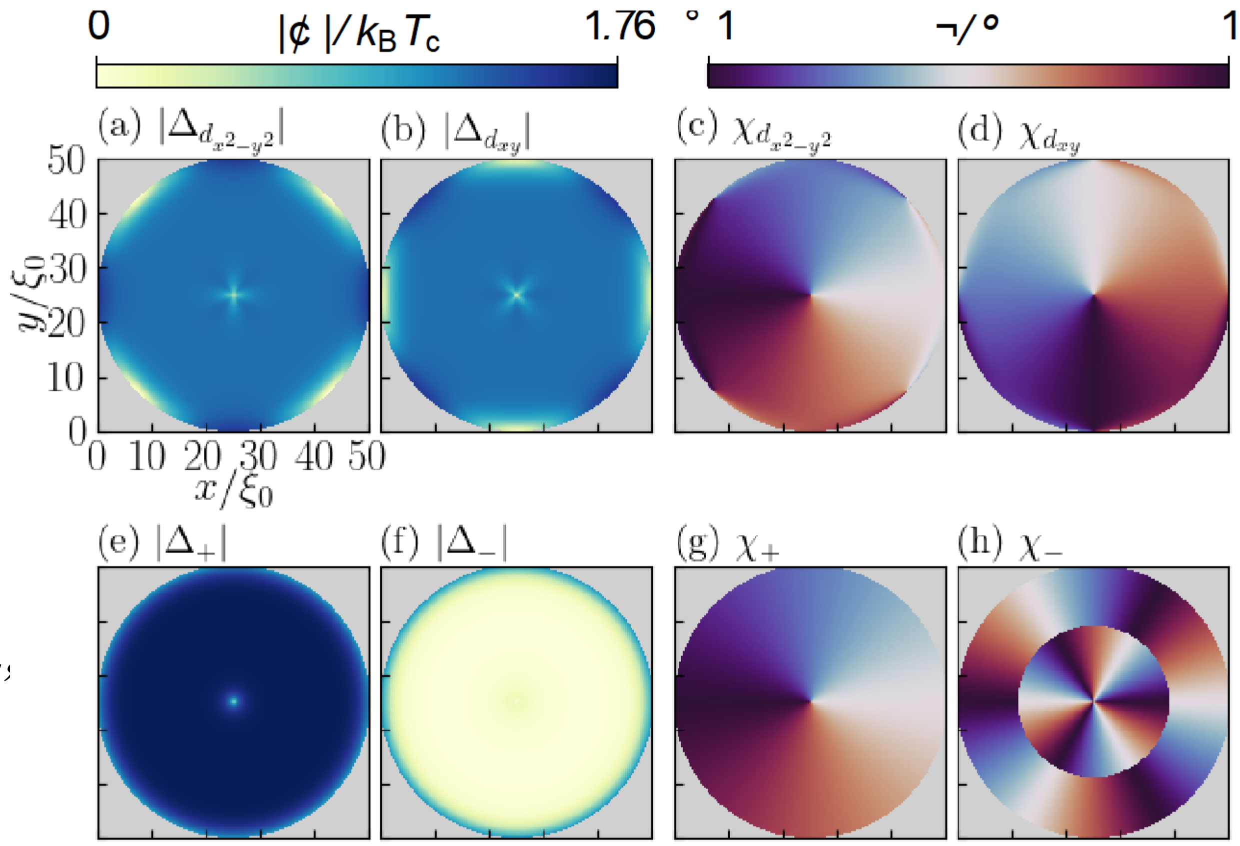


- Edges/Defects: Both chiral solutions may appear
  - Different edges are pair breaking for different parts of Δ

Vortex in Chiral SC

Nodal:
$d(x^2\text{-}y^2)$
$d(xy)$

Chiral:
$\Delta_+ = d + id'$
$\Delta_- = d - id'$



(Abrikosov) vortex: Full 2π phase winding in dominant (Δ_+) chirality component

Holmvall and ABS, PRB **108**, L100506 (2023)

Coreless Vortex

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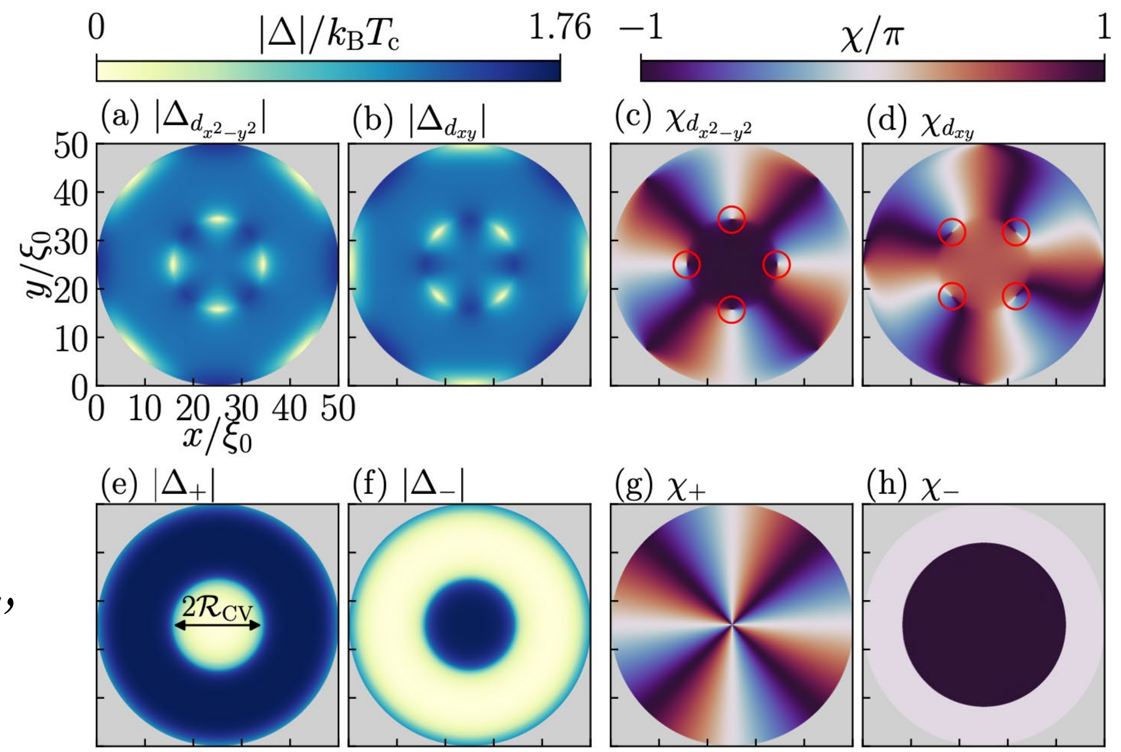
Nodal:
$d(x^2-y^2)$
$d(xy)$

Chiral:
$\Delta_+ = d+\mathrm{i}d'$
$\Delta_- = d-\mathrm{i}d'$



- Phase winding center in chiral component with no amplitude → coreless vortex (CV)
- Fractional vortices in nodal components at DW

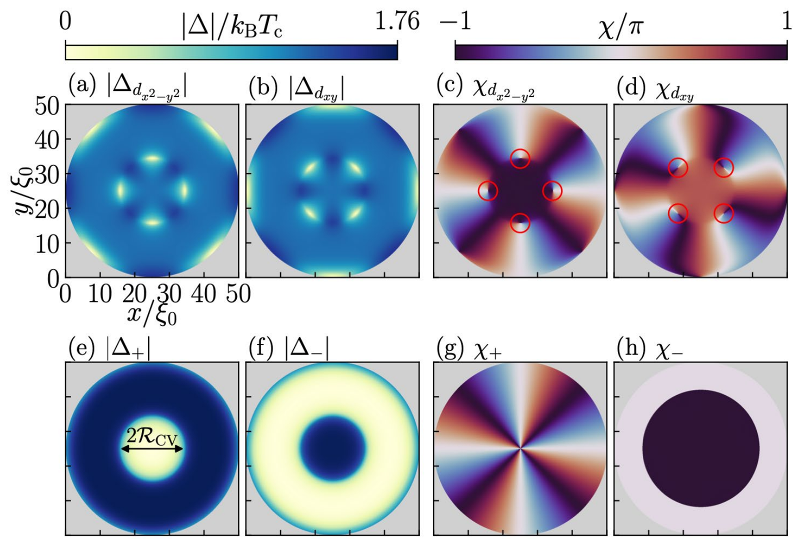
Holmvall and ABS, PRB **108**, L100506 (2023)

Changing Field Direction

Global phase winding in dominant chiral component: m

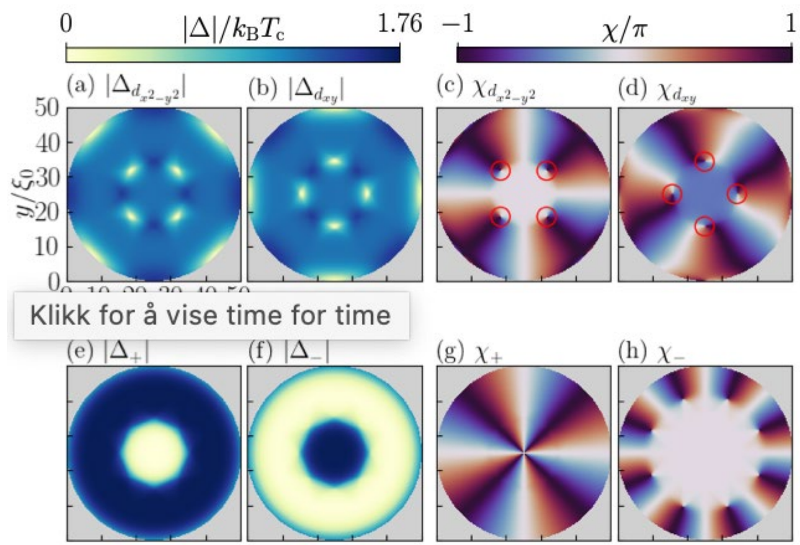
Local phase winding in subdominant chiral component: $p = m + 2\mathrm{N}$

Positive magnetic field ($m < 0$)



Antiparallel CV: m = -4, p = 0

Negative magnetic field ($m > 0$)



Parallel CV: m = 4, p = 8

Sauls and Eschrig, NJP **11**, 075008 (2009); Holmvall and ABS, PRB **108**, L100506 (2023)

Topological Superconductivity

Chiral superconductors

Spin-singlet *d*+i*d'*-wave (spin-triplet *p*+i*p'*-wave) superconductors

Appear often for 2D irreps
Fully gapped bulk
Finite Chern number *N*, set by in-plane phase winding of Δ
Break TRS, preserve at least S_z symmetry
Chiral edge states crossing bulk gap, # = *N*
Host coreless vortices

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Topological Superconductivity

Introduction

Chiral superconductors

Spin-singlet *d*+i*d'*-wave (spin-triplet *p*+i*p'*-wave) superconductors

Spinless superconductors

Majorana fermions

Engineered systems

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"Spinless" *p+ip'* Superconductor

- Spinless superconductor → *p*-wave pairing
- No known intrinsic "spinless" SC
- Multiple proposals for engineered "spinless" *p+ip'* superconductors last ~ 10 years
  - 1D spinless $\Delta \sim k$ (class BDI)
  - 2D spinless $\Delta \sim k_x + ik_y$ (class D)

Can be **topological superconductors**

Topological boundary states are **Majorana Fermions (MFs)**

1D: Localized zero-energy end states

2D: Dispersive edge modes or localized zero-energy vortex states

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Schrödinger, Dirac, and Majorana

Schrödinger (1925)

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi$$

relativistically correct →

Dirac (1928)

$$\sum_{\mu=0}^{3} i\hbar\gamma^\mu\partial_\mu\psi = mc\psi$$

4x4 complex matrices

- Spin-1/2
- Electron & positron (hole)

$$c^\dagger \neq c$$

real equation →

Majorana (1937)

$$\sum_{\mu=0}^{4} i\hbar\tilde{\gamma}^\mu\partial_\mu\psi = mc\psi$$

4x4 imaginary matrices

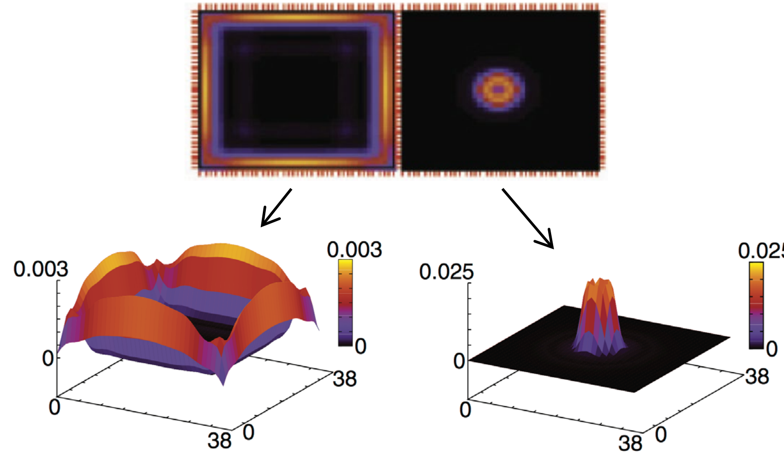
- Particle = Antiparticle: $\gamma^\dagger = \gamma$
- Electron "=" 2 Majorana fermions: $c^\dagger = \gamma_1 + i\gamma_2$

Majorana Fermions

New particle ~ ½ electron

- Emergent particle
- Appears in pairs

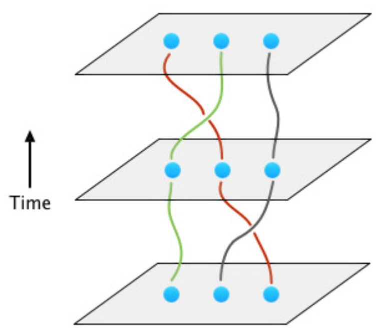
Condensed matter systems



MF on sample edge   MF in vortex core

Non-Abelian statistics in 2D

→ **Robust quantum computation by braiding**



Quantum gate operation
= particle braiding

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Excitations in Superconductors

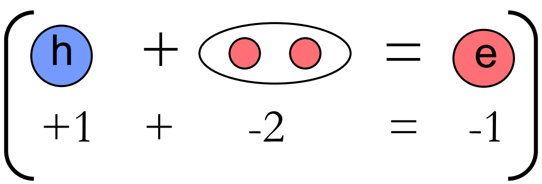
Quasiparticles in a superconductor:

- Part electron and part hole
- Mixed spin-up and spin-down

$$\begin{cases} a = u c_\uparrow^\dagger + v c_\downarrow \\ a^\dagger = u^* c_\uparrow + v^* c_\downarrow^\dagger \end{cases}$$

$$|v_{\mathbf{k}}|^2 = 1 - |u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

→ E = 0 states are Majorana fermions: $\gamma^\dagger = \gamma$ (if we ignore spin)



But …

- Superconductors often have an energy gap
  - Topological SCs have E = 0 boundary states
- E = 0 states are often spin-degenerate (2 Majorana → 1 electron)

→ "Spinless" topological superconductor for MFs

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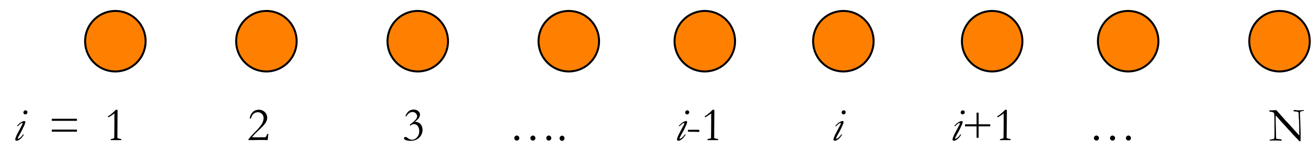
Kitaev's 1D Toy Model

1D chain of spinless electrons with superconducting pairing



$$H = -\mu \sum_i c_i^\dagger c_i - \frac{1}{2} \sum_i t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + \mathrm{H.c.}$$

Chemical potential — Nearest neighbor hopping — Spinless *p*-wave pairing

Kitaev, arXiv:cond-mat/0010440 (2001)

Majorana Basis

$$H = -\mu \sum_i c_i^\dagger c_i - \frac{1}{2} \sum_i t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + \text{H.c.}$$

$$c_i = \frac{1}{2}(\gamma_i^B + i\gamma_i^A)$$

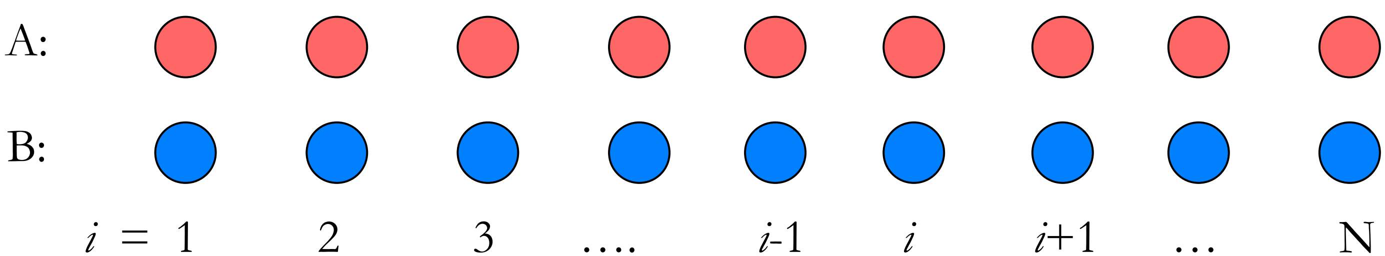
$$(\gamma_i^\alpha)^\dagger = \gamma_i^\alpha$$

$$\{\gamma_i^\alpha, \gamma_j^\beta\} = 2\delta_{ij}\delta_{\alpha\beta}$$

Majorana fermions

Change basis

$$H = -\frac{\mu}{2} \sum_{i=1}^{N} (1 + i\gamma_i^B \gamma_i^A) - \frac{i}{4} \sum_{i=1}^{N-1} \left[(\Delta + t)\gamma_i^B \gamma_{i+1}^A + (\Delta - t)\gamma_i^A \gamma_{i+1}^B\right]$$



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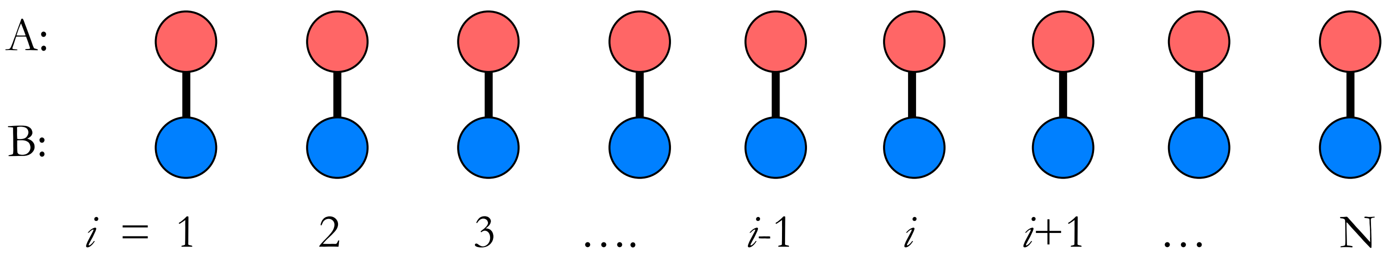
Trivial Phase

$$H = -\frac{\mu}{2}\sum_{i=1}^{N}(1 + i\gamma_i^B\gamma_i^A) - \frac{i}{4}\sum_{i=1}^{N-1}\left[(\Delta + t)\gamma_i^B\gamma_{i+1}^A + (\Delta - t)\gamma_i^A\gamma_{i+1}^B\right]$$

Topological trivial phase: $\Delta = t = 0$, $\mu < 0$



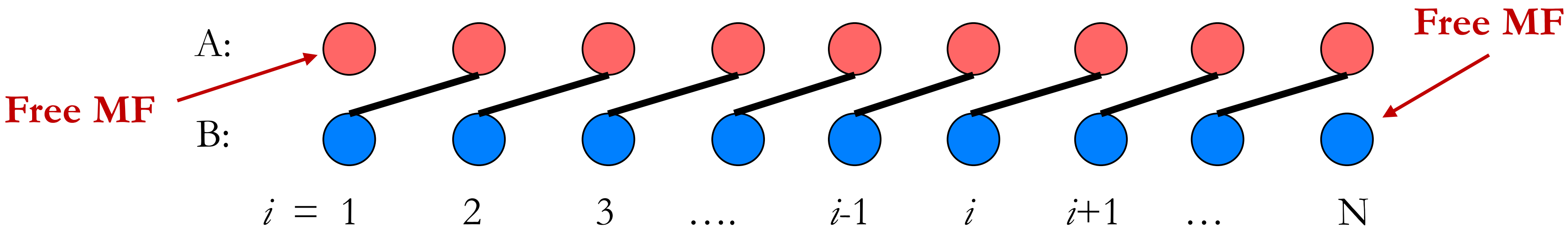
Unique ground state

- Vacuum state for electrons
- Finite bulk gap (|μ| lowest excitation energy)

Non-Trivial Phase

$$H = -\frac{\mu}{2}\sum_{i=1}^{N}(1 + i\gamma_i^B\gamma_i^A) - \frac{i}{4}\sum_{i=1}^{N-1}\left[(\Delta + t)\gamma_i^B\gamma_{i+1}^A + (\Delta - t)\gamma_i^A\gamma_{i+1}^B\right]$$

Topological non-trivial phase: $\mu = 0$, $\Delta = -t \neq 0$



$$\begin{cases} d_i = \frac{1}{2}(\gamma_{i+1}^A + i\gamma_i^B) \\ f = \frac{1}{2}(\gamma_1^A + i\gamma_N^B) \end{cases} \Rightarrow H = t\sum_i^{N-1}(d_i^\dagger d_i - \frac{1}{2})$$

Degenerate ground state

- Finite bulk gap
- Zero-energy MFs at end points

Majorana Fermions in BdG

How can we get "½ electron" in the BdG formalism?

Never if $\epsilon_{\mathbf{k}} \sum_{\sigma} c^{\dagger}{}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \left[\bar{\Delta} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^{\dagger}{}_{\mathbf{k}\uparrow} c^{\dagger}{}_{-\mathbf{k}\downarrow} \Delta\right] = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right) \begin{bmatrix} \epsilon_{\mathbf{k}} & \Delta \\ \bar{\Delta} & -\epsilon_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix}$

→ Not in spin-degenerate (e.g. chiral *p*+i*p*' or *d*+i*d*') superconductors

But if $\tilde{\mathcal{H}} = \mathbf{a}^{\dagger}_{\mathbf{k}} \begin{pmatrix} \varepsilon(\mathbf{k}) & \hat{\Delta}(\mathbf{k}) \\ \hat{\Delta}^{\dagger}(\mathbf{k}) & -\varepsilon(\mathbf{k}) \end{pmatrix} \mathbf{a}_{\mathbf{k}}$ $\quad \left[\mathbf{a}_{\mathbf{k}} = (a_{\mathbf{k}\uparrow}, a_{\mathbf{k}\downarrow}, a^{\dagger}{}_{-\mathbf{k}\uparrow}, a^{\dagger}{}_{-\mathbf{k}\downarrow})\right]$

1 electron represented by 2 eigenvector components

→ MF if E=0 eigenstate has no spatial overlap with other states

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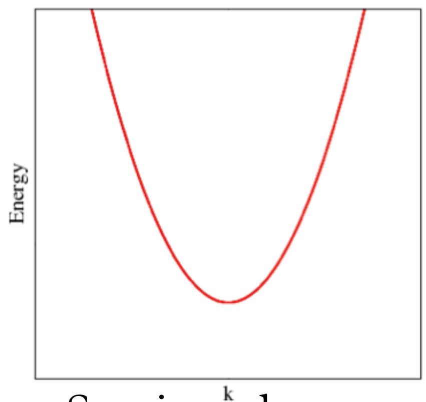
SOC Semiconductors

Spin-orbit coupled (SOC) semiconductor + magnetic field



Semiconductor, spin degenerate

Conventional s-wave pairing

4x4 BdG description needed due to SOC + Zeeman field

Spinless *p*+i*p*' superconductor with MFs if $|V_z| > |\Delta|$

2D: Sau *et al.* PRL **104**, 040502 (2010). 1D: Lutchyn *et al.* PRL **105**, 077001 (2010), Oreg *et al.* PRL **104**, 077002 (2010)

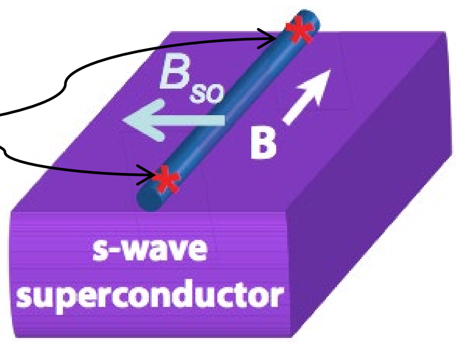
Experimental Hunt in Nanowires

1D InSb nanowire
(Semiconductor with strong SOC)

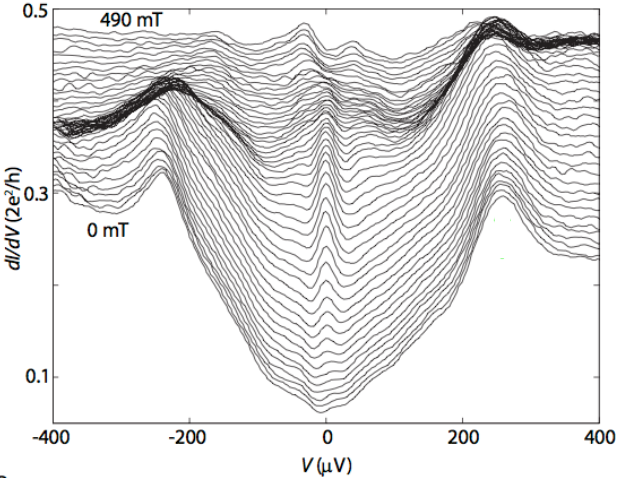
\+ *s*-wave superconductor

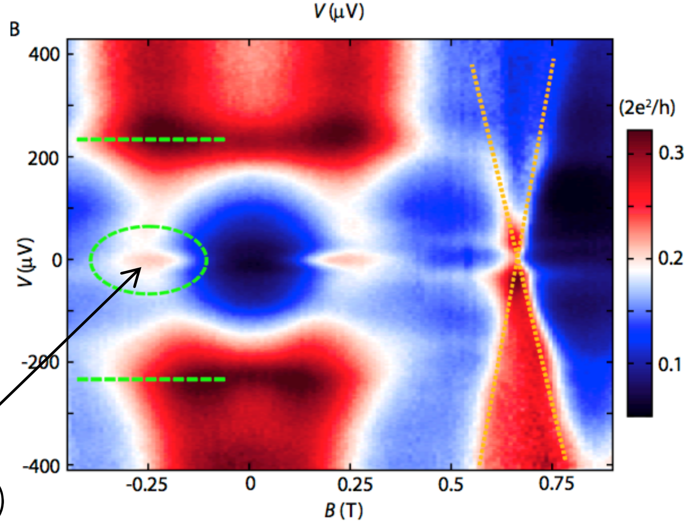
\+ Magnetic field

MFs?



Conductance through wire





MFs (?)

Mourik *et al.*, Science **336**, 1003 (2012)

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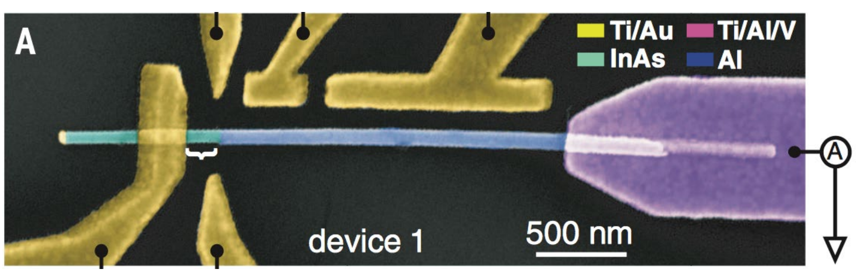
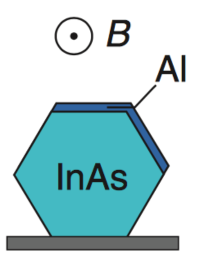
Nanowires with Hard Gaps

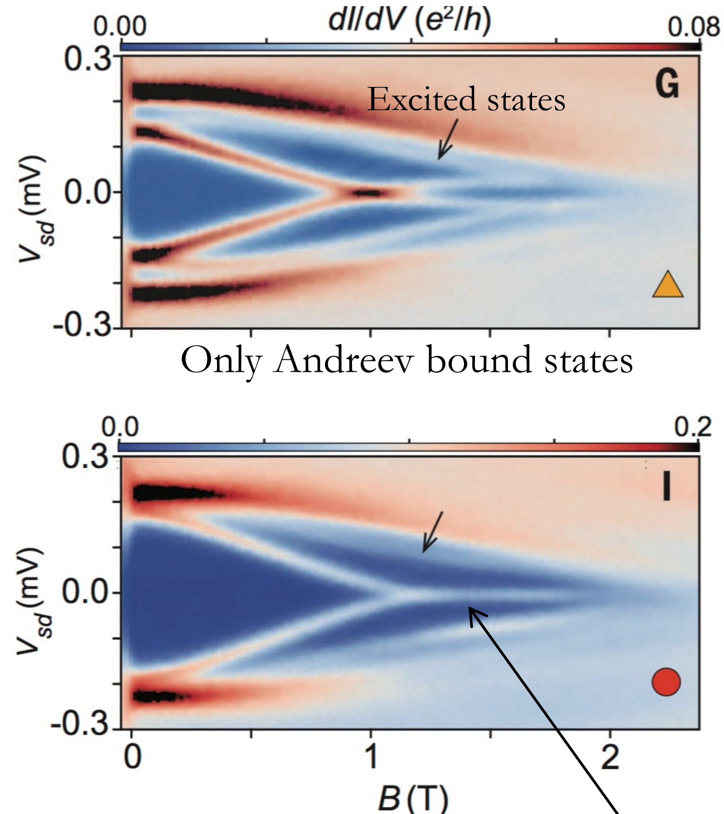
1D InAs nanowire
+ Al superconductor
+ Magnetic field

} Hard SC gap





Conductance at different gate biases



Topological phase with MF (?)

Deng *et al.*, Science **354**, 1557 (2016)

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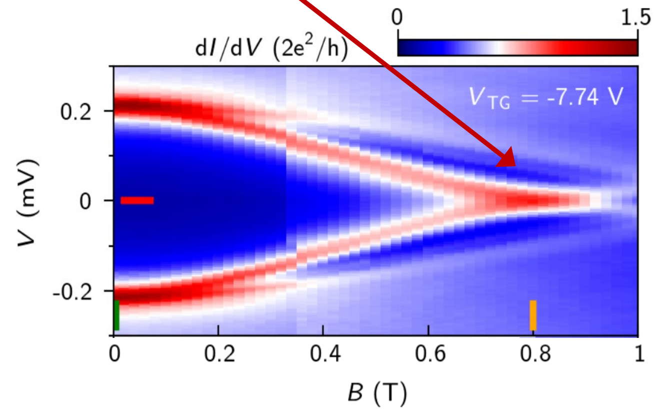
Is it a Majorana?

How to distinguish MFs?

- Stable zero-energy peak
- Quantized conductance
- Bulk gap closing
- …

Not unique to MBS

MFs (?)



Problem:

Interfaces/edges/impurities often host trivial (accidental) zero-energy Andreev bound states (ABS)

Zhang *et al.*, arXiv:2101.11456, see also Prada *et al.*, Nat. Rev. Phys. 2, 575 (2020)

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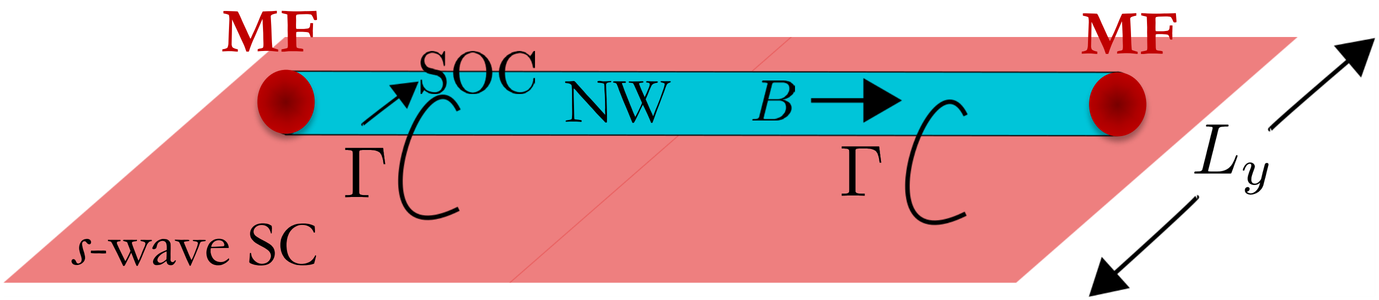
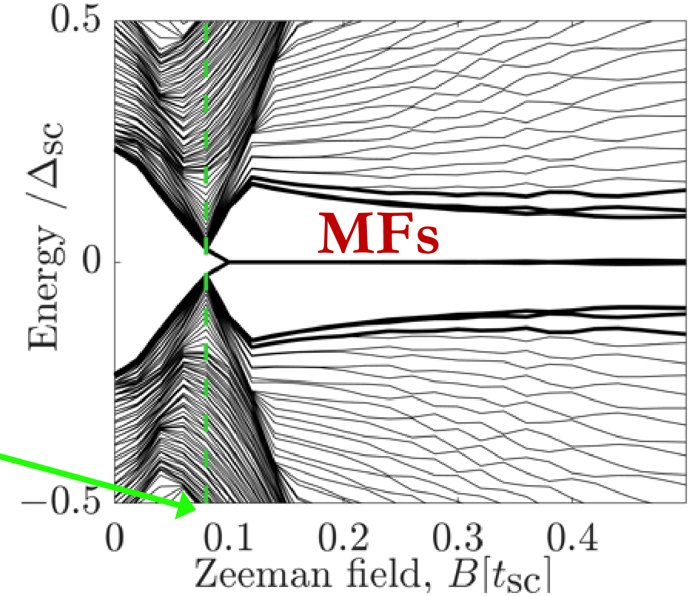
Nanowire + Superconductor

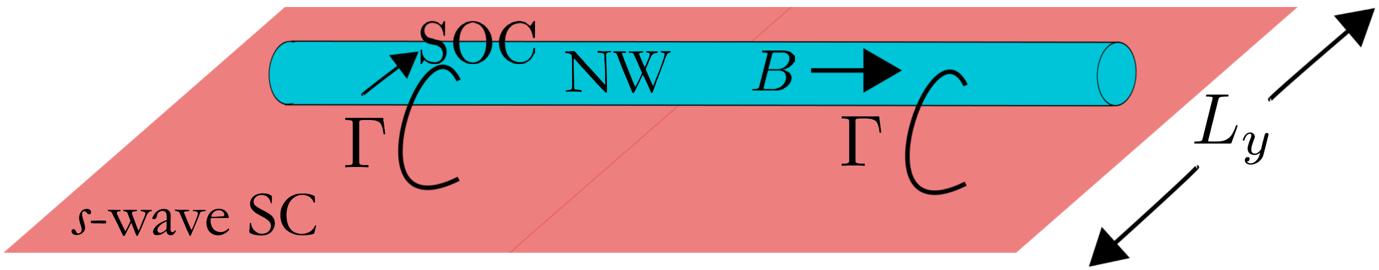


$$H = H_{\mathrm{SC}} + H_{\mathrm{NW}} + H^{\Gamma}_{\mathrm{SC-NW}}$$

MBS beyond topological phase transition (TPT) at B_c

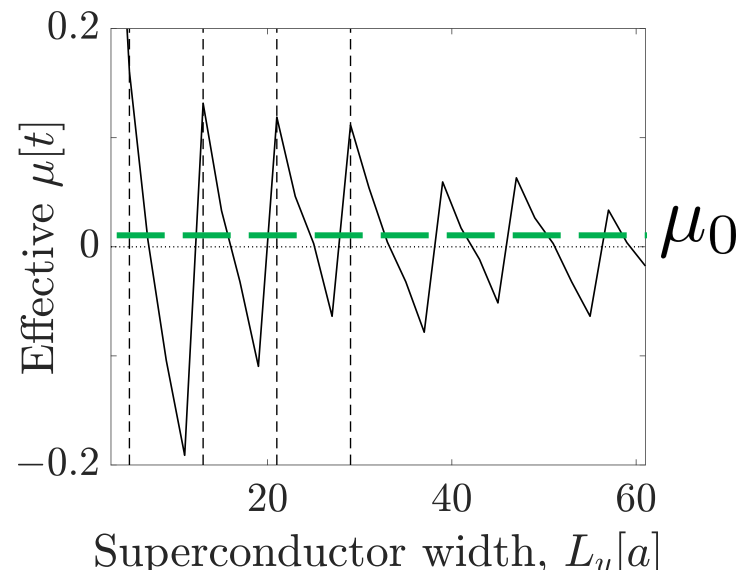


Nanowire + Superconductor



$$H = H_{\rm SC} + H_{\rm NW} + H^{\Gamma}_{\rm SC-NW}$$

Heavily modified effective chemical potential (and SOC) in NW



Awoga, Cayao, and ABS, PRL **123**, 117001 (2019), see also Reeg et al. PRB 97, 165425 (2018)

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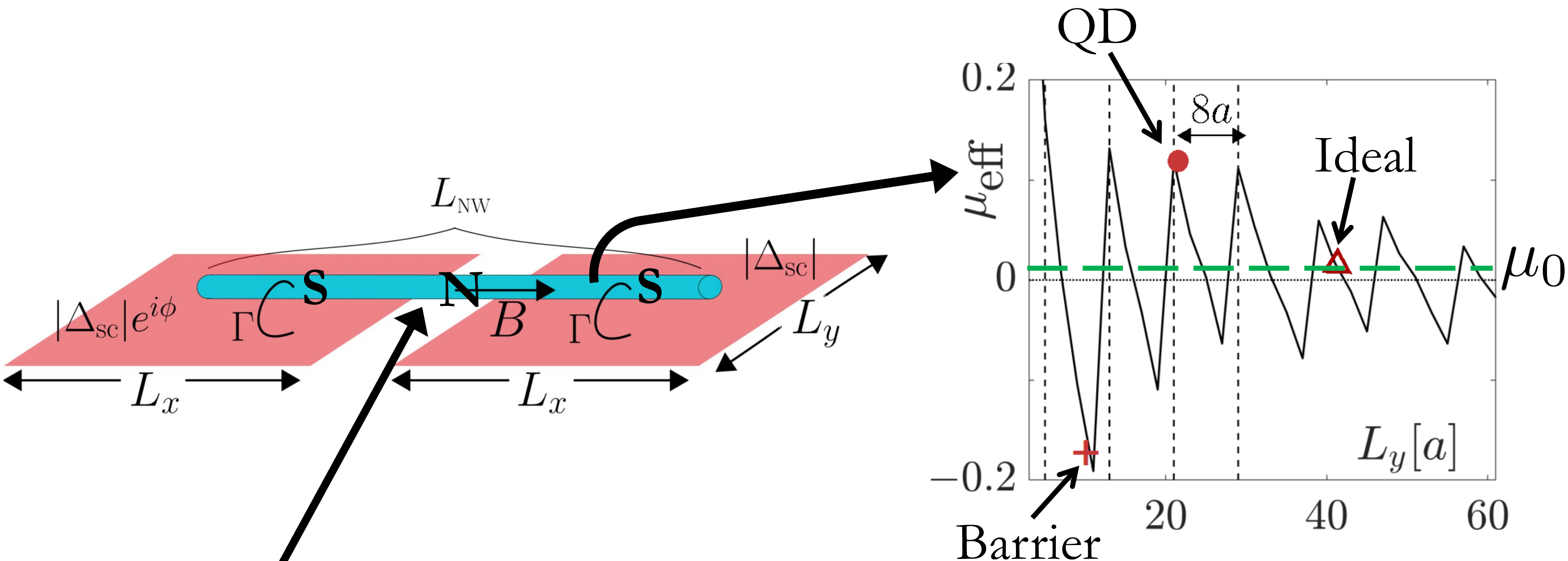
Short SNS Junction



Quantum dot (QD) or barrier emerges spontaneously in short NW junctions

Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

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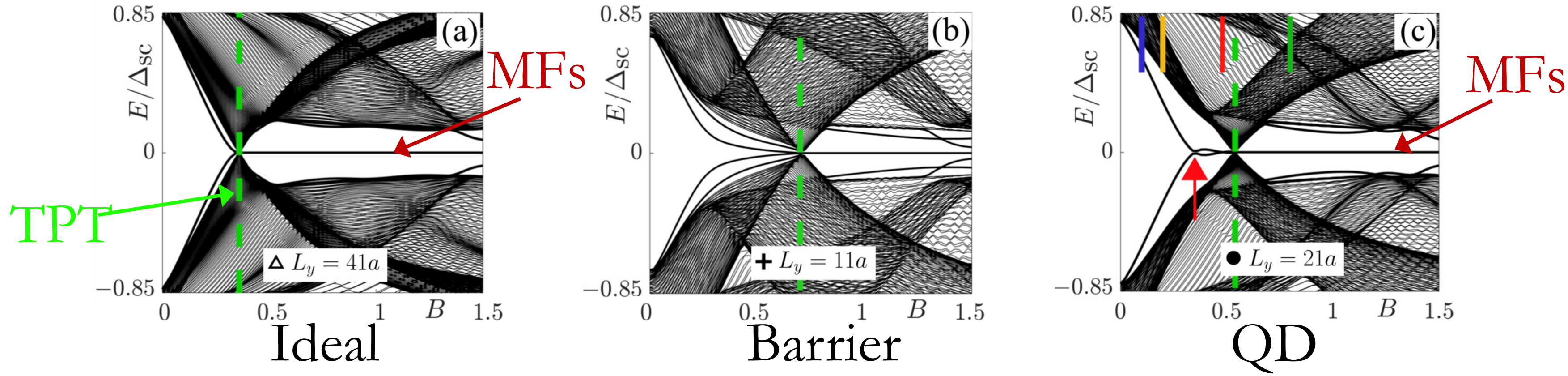
False MFs in QD Regime

Energy spectrum of junction



Zero-energy QD states before TPT

→ **false MFs**

Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

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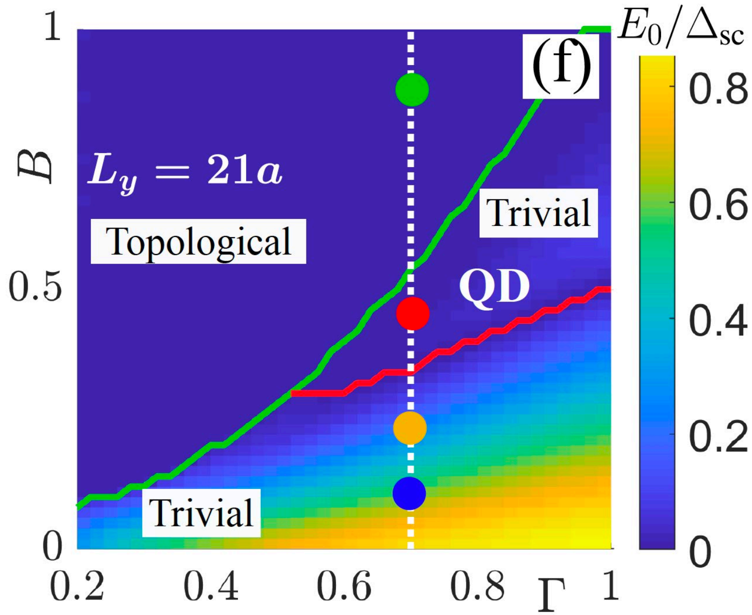
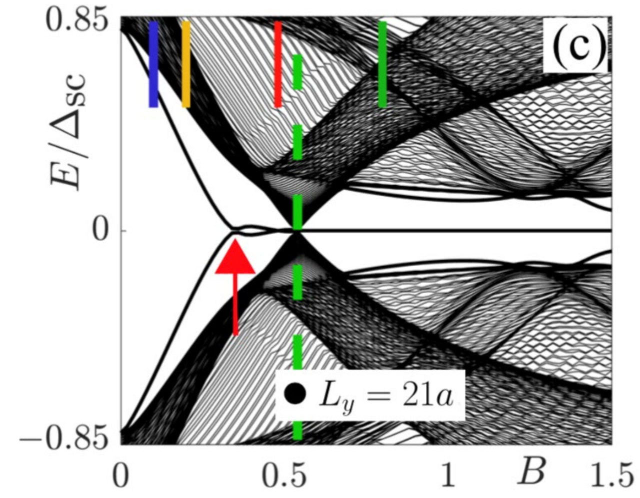
False MFs in QD Regime





Zero-energy (trivial) QD states
always in strong coupling regime

Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

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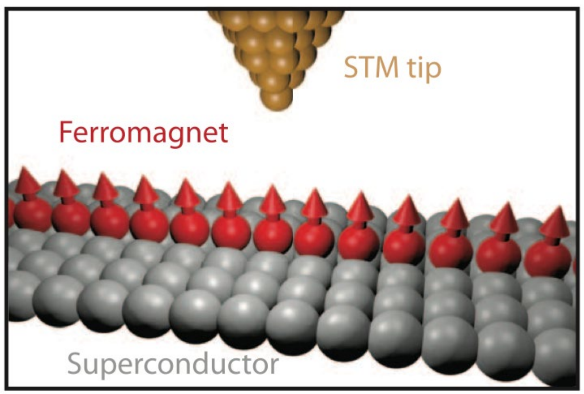
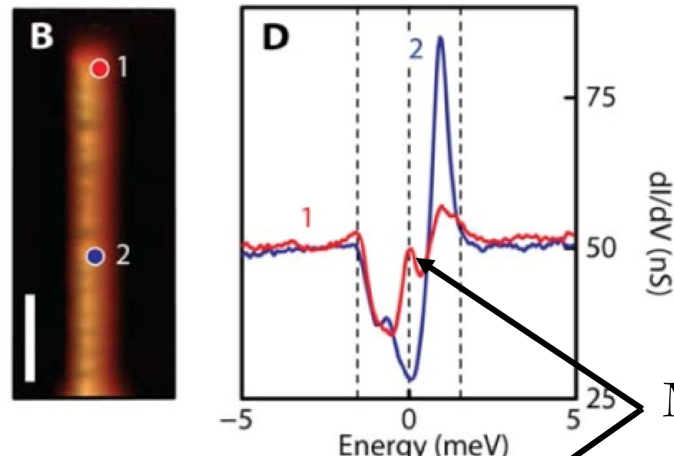
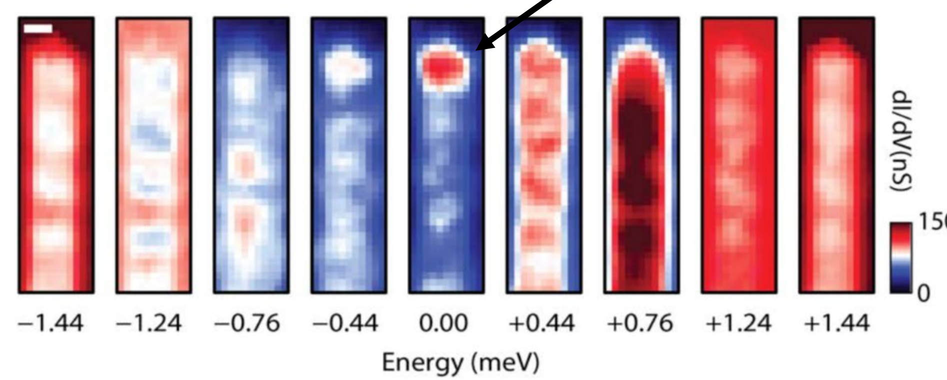
Hunt with Magnetic Atoms

Pb substrate
(SC with strong SOC)
+ Fe ad-atoms





MFs (?)



Also: MFs with predicted spin-polarization

Nadj-Perge *et al.*, Science **346**, 602 (2014), Jeon *et al.*, Science **358**, 772 (2017)

Magnetic Atoms on Superconductors

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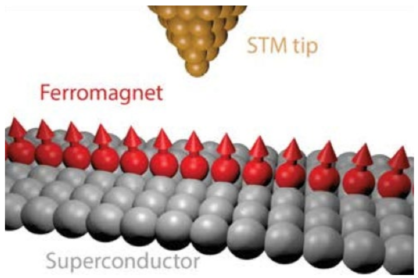
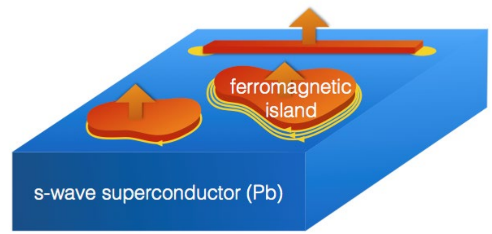
Magnetic atoms on a SOC superconductor





$$\mathcal{H} = \mathcal{H}_{kin} + \mathcal{H}_{SO} + \mathcal{H}_{sc} + \mathcal{H}_{V_z}$$

SOC superconductor

$$\mathcal{H}_{kin} = -t \sum_{\langle \mathbf{i},\mathbf{j}\rangle,\sigma} c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} - \mu \sum_{\mathbf{i},\sigma} c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{i}\sigma}$$

$$\mathcal{H}_{SO} = -\frac{\lambda}{2} \sum_{\mathbf{i}} \Big[(c^\dagger_{\mathbf{i}-\hat{x}\downarrow} c_{\mathbf{i}\uparrow} - c^\dagger_{\mathbf{i}+\hat{x}\downarrow} c_{\mathbf{i}\uparrow}) + i(c^\dagger_{\mathbf{i}-\hat{y}\downarrow} c_{\mathbf{i}\uparrow} - c^\dagger_{\mathbf{i}+\hat{y}\downarrow} c_{\mathbf{i}\uparrow}) + \text{H.c.} \Big]$$

$$\mathcal{H}_{sc} = \sum_{\mathbf{i}} \Delta_{\mathbf{i}} (c^\dagger_{\mathbf{i}\uparrow} c^\dagger_{\mathbf{i}\downarrow} + \text{H.c.})$$

Magnetic atoms on sites **a**

(to 1st approximation)

$$\mathcal{H}_{V_z} = -\sum_{\mathbf{a},\sigma,\sigma'} (V_z(\mathbf{a}) \hat{\mathbf{n}} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} c^\dagger_{\mathbf{a}\sigma} c_{\mathbf{a}\sigma'}$$

Nadj-Perge *et al.*, Science **346**, 6209 (2014), Li *et al.*, Nat. Commun. **7**, 12297 (2016)

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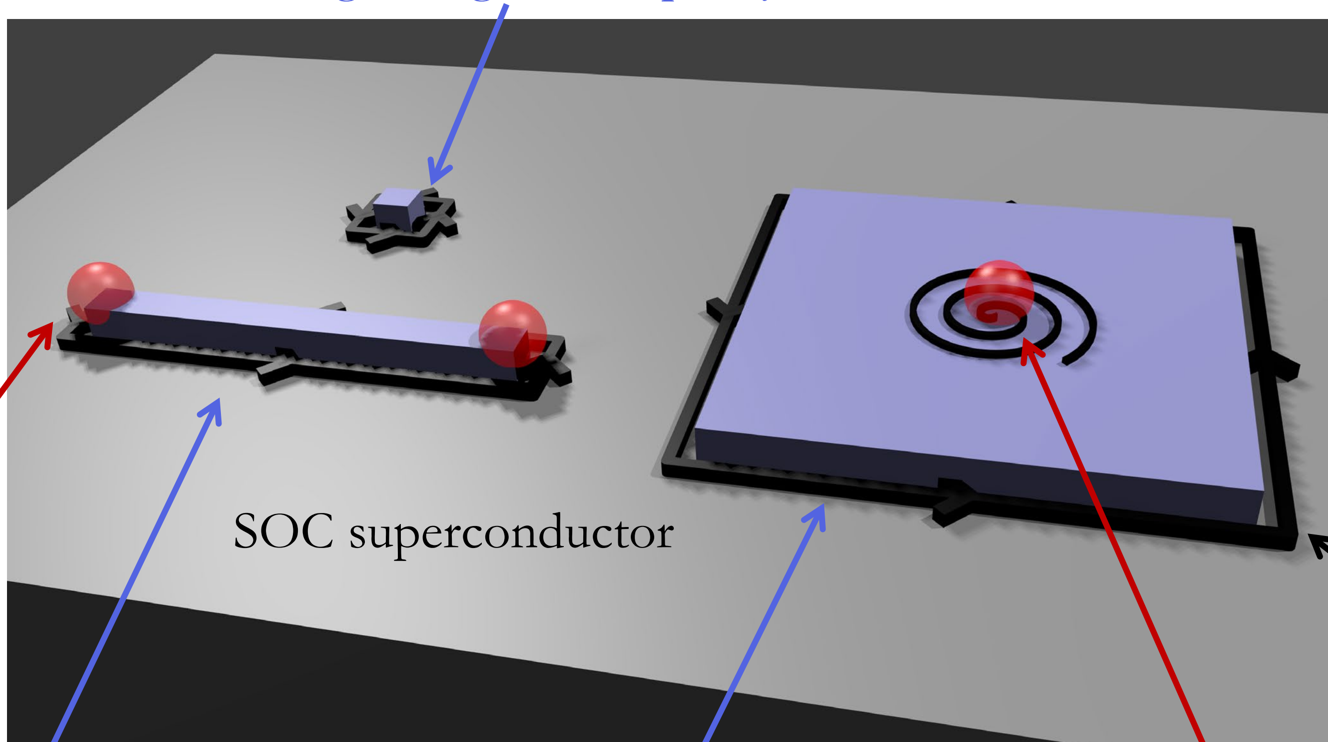
Flexible Setup

Self-consistent solution for the superconducting order parameter

$\left[\Delta_{\mathbf{i}} = -V_{sc}\langle c_{\mathbf{i}\downarrow} c_{\mathbf{i}\uparrow}\rangle\right]$

Single magnetic impurity



MF wire
end state

1D ferromagnetic wire

2D ferromagnetic island

MF vortex
core state

Circulating
currents

PRL 115, 116602
(2015)

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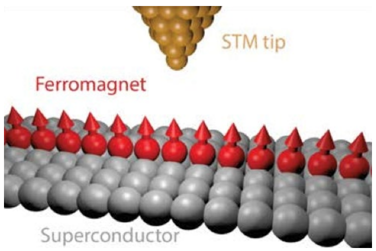
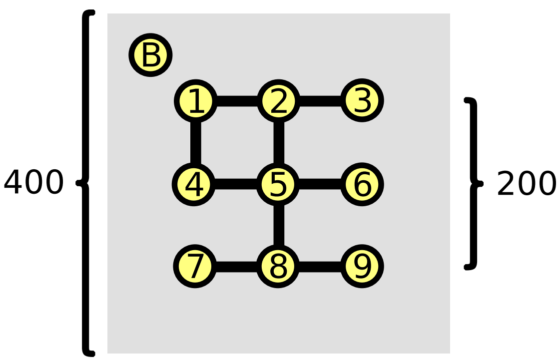
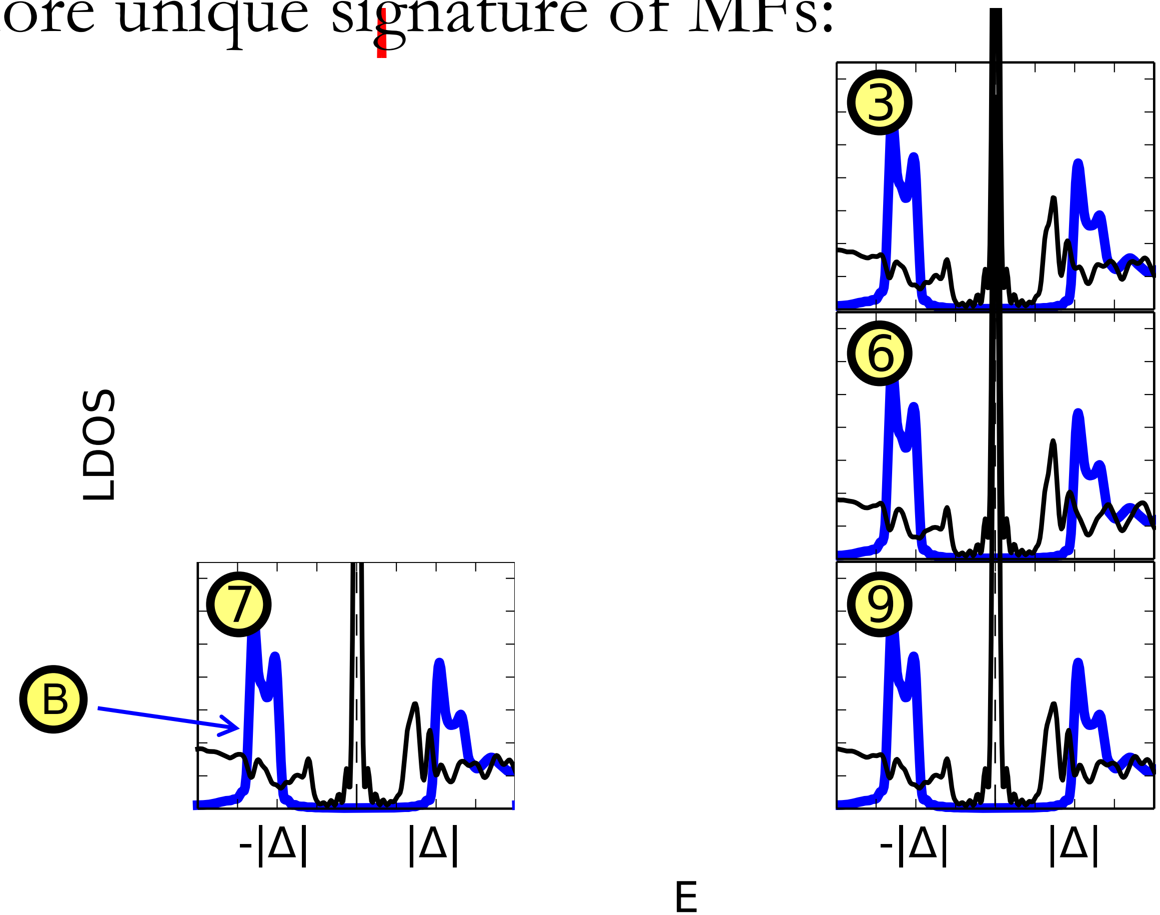
Ferromagnet Atom Chain Network

Wire network for more unique signature of MFs:







Zero-energy MFs at odd-wire junctions (black)

No subgap states at even-wire junctions (red)

Björnson and ABS, PRB **94**, 100501(R) (2016)

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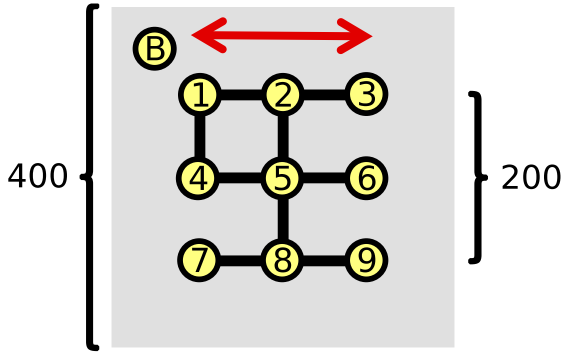
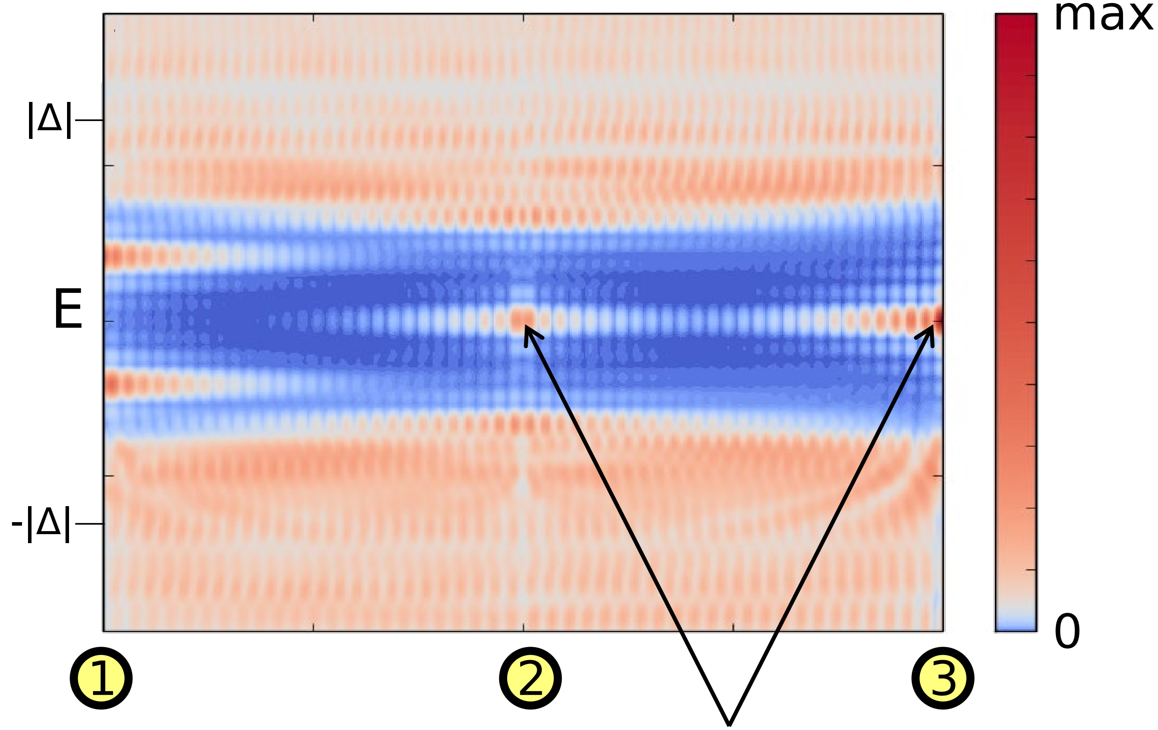
Odd- and Even-Wire Junctions



LDOS along upper wire segment



Only clear subgap states are MFs at odd-wire junctions

MF wire end states

Björnson and ABS, PRB **94**, 100501(R) (2016)

Topological Superconductivity

Spinless superconductors

Prototype: Kitaev model for 1D spinless SC
Materials: SOC + magnetism + *s*-wave SC
Majorana fermion:

- Non-local, "½ electron"
- Zero-energy topological boundary state in spinless SCs
    - Protected by energy gap
    - Note: not all zero-energy states in SCs are MFs

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Summary

- Introduction to superconductivity
  - BCS, BdG, group theory

- Topological superconductivity
  - **Chiral superconductvity: *p*+i*p'*- and *d*+i*d'*-wave symmetry**
    - Appears often in 2D irreps
    - Topology set by Chern number (winding of order parameter)
    - Chiral (electronic) edge states

  - **Spinless topological superconductivity**
    - SOC + magnetism + *s*-wave superconductivity
    - Zero-energy edge state = Majorana fermion

General Hamiltonian

General Hamiltonian: $\mathcal{H}=\sum_{\mathbf{k},s}\varepsilon(\mathbf{k})a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s}$

$$+\frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',s_1,s_2,s_3,s_4}V_{s_1s_2s_3s_4}(\mathbf{k},\mathbf{k}')a^\dagger_{-\mathbf{k}s_1}a^\dagger_{\mathbf{k}s_2}a_{\mathbf{k}'s_3}a_{-\mathbf{k}'s_4}$$

Mean-field order : $\Delta_{ss'}(\mathbf{k})=-\sum_{\mathbf{k}',s_3,s_4}V_{s'ss_3s_4}(\mathbf{k},\mathbf{k}')\langle a_{\mathbf{k}'s_3}a_{-\mathbf{k}'s_4}\rangle$

→ $\tilde{\mathcal{H}}=\sum_{\mathbf{k},s}\varepsilon(\mathbf{k})a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s}+\frac{1}{2}\sum_{\mathbf{k},s_1,s_2}[\Delta_{s_1s_2}(\mathbf{k})a^\dagger_{\mathbf{k}s_1}a^\dagger_{-\mathbf{k}s_2}$

$$-\Delta^*_{s_1s_2}(-\mathbf{k})a_{-\mathbf{k}s_1}a_{\mathbf{k}s_2}]$$

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Matrix Formulation

4-component notation (Nambu): $\mathbf{a_k} = (a_{\mathbf{k}\uparrow}, a_{\mathbf{k}\downarrow}, a^{\dagger}_{-\mathbf{k}\uparrow}, a^{\dagger}_{-\mathbf{k}\downarrow})^{T}$

$$\rightarrow \tilde{\mathcal{H}} = \mathbf{a}_{\mathbf{k}}^{\dagger} \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \hat{\Delta}(\mathbf{k}) \\ \hat{\Delta}^{\dagger}(\mathbf{k}) & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix} \mathbf{a}_{\mathbf{k}}$$

Spin-singlet pairing: $\hat{\Delta}(\mathbf{k}) = i\hat{\sigma}_y \psi(\mathbf{k}) = \begin{bmatrix} 0 & \psi(\mathbf{k}) \\ -\psi(\mathbf{k}) & 0 \end{bmatrix}$

$\left(\psi(\mathbf{k})[c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow} - c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}]\right)$

ψ even function of **k**

Spin-triplet pairing: $\hat{\Delta}(\mathbf{k}) = i(\mathbf{d}(\mathbf{k})\cdot\hat{\boldsymbol{\sigma}})\hat{\sigma}_y$

$$= \begin{bmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & d_z(\mathbf{k}) \\ d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{bmatrix}$$

d vector odd function of **k**

$$\begin{pmatrix} m_z = 0: & d_z(\mathbf{k})[c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow} + c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}] \\ m_z = 1: & [-d_x(\mathbf{k}) + id_y(\mathbf{k})]c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\uparrow} \end{pmatrix}$$

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General Solution

QP energy (eigenvalue): $E_{\mathbf{k}} = \sqrt{\varepsilon(\mathbf{k})^2 + |\psi(\mathbf{k})|^2}$

$$E_{\mathbf{k}} = \sqrt{\varepsilon(\mathbf{k})^2 + |\mathbf{d}(\mathbf{k})|^2 \pm \mathbf{q}(\mathbf{k})|}$$

$$\left(\begin{array}{c} \hat{\Delta}\hat{\Delta}^{\dagger} = |\mathbf{d}|^2 \hat{\sigma}_0 + \mathbf{q}\cdot\hat{\boldsymbol{\sigma}} \\ \mathbf{q} = i(\mathbf{d}\times\mathbf{d}^*) \\ \text{Finite } \mathbf{q} = \text{non-unitary} \end{array} \right)$$

Self-consistency equation, linear close to T_c:

$$v\Delta_{s_1s_2}(\mathbf{k}) = -\sum_{s_3s_4} \langle V_{s_2s_1s_3s_4}(\mathbf{k},\mathbf{k}')\Delta_{s_3s_4}(\mathbf{k}')\rangle_{\mathbf{k}'}$$

$$\frac{1}{v} = N(0)\int_0^{\varepsilon_c} d\varepsilon \frac{\tanh\left[\frac{\beta_c \varepsilon(k)}{2}\right]}{\varepsilon(\mathbf{k}} = \ln(1.14\beta_c\varepsilon_c)$$

- Largest eigenvalue gives T_c
- Eigenfunction (Δ) belongs to irreducible representation (irrep) of symmetry group

→ Possible SC symmetries belong to **irreps of symmetry group of H**

→ SC state always breaks U(1), can also break

– Crystal lattice, spin-rotation, time-reversal, … symmetries

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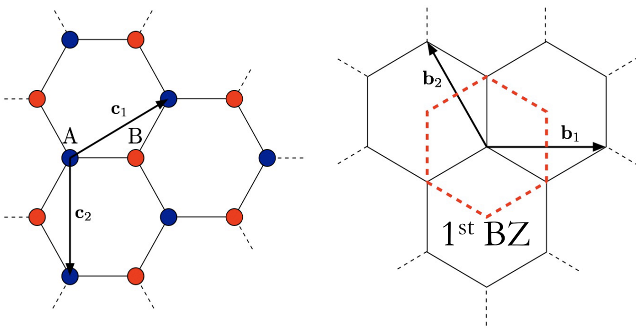
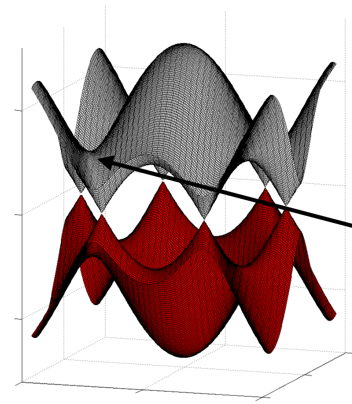
Graphene a *d*+i*d'* SC?

Honeycomb lattice



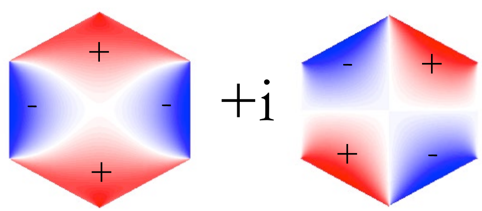
Band structure with van Hove singularities



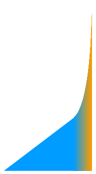
Diverging DOS
→ favorable for SC

Pairing from repulsive interactions

- Strong interactions [1]
- Perturbative RG [2]
- Functional RG [3]



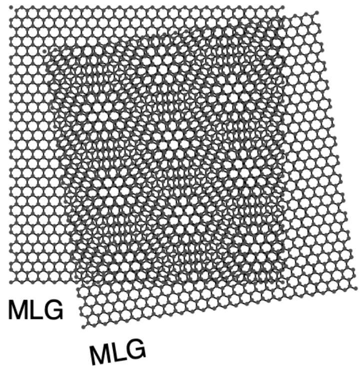
$d(x^2-y^2)$ +i $d(xy)$



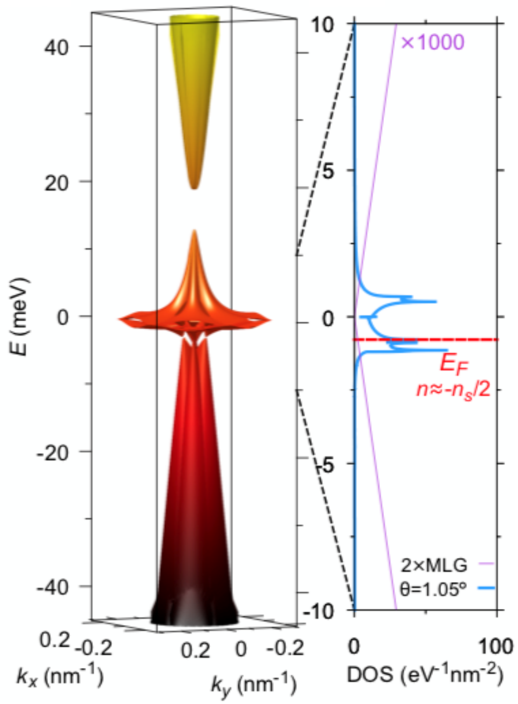
[1]: ABS and Doniach, PRB **75**, 134512 (2007), [2]: Nandkishore *et al.*, Nat. Phys. **8**, 158 (2012), [3]: Kiesel *et al.*, PRB **86**, 020507 (2012)

Twisted Bilayer Graphene

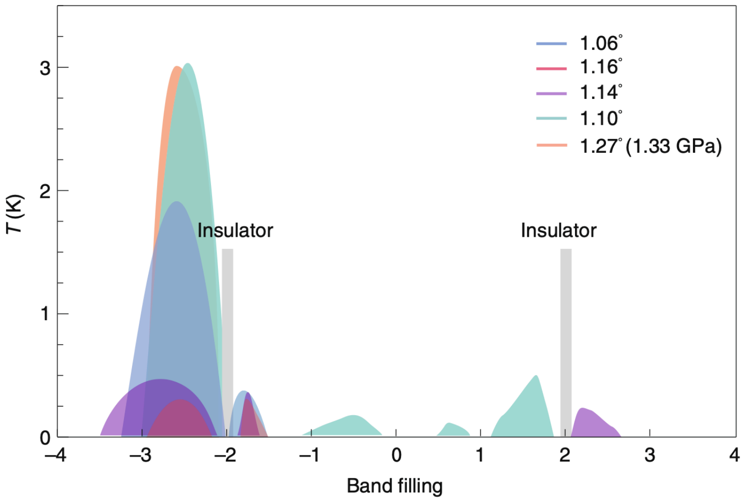
Supercell moiré pattern



Small "magic" angles → **low energy flat bands**



Superconducting domes throughout moiré flat band



Bistritzer&MacDonald, PNAS 108, 12233 (2011); Cao et al, Nature 556, 43 (2018); Nature 556, 80 (2018); Balents et al, Nat. Phys. 16, 725 (2020)